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H_3^+ WZNW model from Liouville field theory

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Abstract: There exists an intriguing relation between genus zero correlation functions in the H_3^+ WZNW model and in Liouville field theory. We provide a path integral derivation of the correspondence and then use our new approach to generalize the relation to surfaces of arbitrary genus g. In particular we determine the correlation functions of N primary fields in the WZNW model explicitly through Liouville correlators with $N+2g-2$ additional insertions of certain degenerate fields. The paper concludes with a list of interesting further extensions and a few comments on the relation to the geometric Langlands program.

Keywords: Conformal Field Models in String Theory, Integrable Equations in Physics.

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1. Introduction

The H_3^+ Wess-Zumino-Novikov-Witten (WZNW) model has received considerable attention as an interesting non-rational conformal field theory [1, 2] and as the Euclidean version of AdS_3 [3]. The 3-dimensional target space of the theory may be parametrized by coordinates γ and $\bar{\gamma}$ of its 2-dimensional boundary along with some radial coordinate ϕ. As for any Anti-de Sitter (AdS) geometry, the latter is particularly interesting. Physically, it should be regarded as a very close relative of the Liouville direction in 2-dimensional string theory.

It was long observed [1] that the 3-point functions of the H_3^+ WZNW model coincide with those of Liouville field theory [4 − 6] up to some simple kinematical factors that are determined by the SL(2,ℂ) symmetry of the H_3^+ background. In particular, all the highly nontrivial (non-perturbative) curvature dependence of the H_3^+ 3-point couplings is inherited from the stringy corrections of Liouville theory. A very remarkable generalization of this fact was discovered in [7]: Ribault and Teschner determined genus zero correlators for any number N of WZNW primaries in terms of certain $2N-2$ point functions in Liouville theory. Their proof relies on a relation [8] between differential equations of both theories, namely the Knizhnik-Zamolodchikov (KZ) equations for WZNW models and the Belavin-Polyakov-Zamolodchikov (BPZ) equations for Liouville correlators, along with descent relations provided by the explicit knowledge of the 3-point functions. The relation between Liouville theory and the H_3^+ model was further explored in [9–11].

The aim of this paper is twofold. To begin with, we shall re-derive the Ribault-Teschner correspondence for genus zero correlation functions using a rather elegant path integral computation instead of heavy algebraic manipulations. Our derivation provides a completely new view on the map between WZNW and Liouville primaries and on the necessity to introduce $N-2$ further degenerate fields in the Liouville correlation function. The simplicity of our derivation opens the way to various generalizations. Among them is the extension of the correspondence to correlators on higher genus surfaces. In fact, using essentially the same ideas as for the tree-level computation, we shall derive a precise expression for N-point correlators of WZNW primaries on any closed Riemann surface in terms of Liouville field theory. On the Liouville side, the construction involves correlation functions with $N+2g-2$ additional insertions of degenerate fields. Further extensions, e.g. to the case with world-sheets with boundaries or target space groups of rank $r>1$ will be briefly discussed at the end of this work.

Let us now describe the contents of each section and our main results in a bit more detail. As we have mentioned, we shall start our analysis by giving a new derivation of the Ribault-Teschner correspondence on the sphere. For the convenience of the reader, we also include a few comments on the relation between differential equations. Section 3 contains the generalization to the torus. For genus $g=1$, all the special functions involved in the formulation of the correspondence can be easily expressed in terms of Jacobi's θ function. This makes our formulas particularly easy to deal with. In particular, we shall be able to demonstrate explicitly how our correspondence intertwines between the relevant differential equations. Correlation functions of Liouville theory on the torus satisfy an extension of the BPZ differential equation [12] that involves a derivative with respect to the modular parameter τ of the torus in addition to derivatives with respect to the insertion points of fields. On the WZNW side, the story is a bit more complicated. In fact, the usual WZNW correlators do not obey KZ-type equations. The problem arises from the zero modes of currents which cannot be written as differential operators acting on correlation functions. This issue was resolved by Bernard in [13], who had the idea to introduce an additional dependence on the choice of some group element. The latter parametrizes possible twisted boundary conditions for currents along the β-cycle of the torus. Not all these *dim* G parameters are actually needed for the formulation of KZ-type equations on the torus. The minimal number of required extra parameters is given by the rank of G rather than its dimension. In our case this means that one extra twist parameter λ is sufficient. Once such an extension of WZNW correlators is taken into account, they satisfy the Knizhnik-Zamolodchikov-Bernard (KZB) differential equations. By our correspondence, these are correctly related to the genus $g=1$ BPZ equations.

Section 4 contains the main new formula of this work, namely a construction of N-

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point WZNW correlators on a closed Riemann surface of genus g from correlation functions in Liouville field theory, see eq.$\tilde{(}$4.14). As we stated before, the latter involves $N + 2g - 2$ insertions of Liouville degenerate fields in addition to the N primary fields. It is instructive to compare the parameter spaces on both sides of this correspondence. Obviously, the correlation functions of the WZNW model must be extended by introducing extra parameters generalizing the role of the modulus λ we discussed at length in the previous paragraph. It turns out that on a surface of genus $g \geq 1$ we need $2g - 1$ new complex coordinates. For fixed surface moduli τ and insertion points $z_\nu, \nu = 1, \ldots, N$, an N-point function of WZNW primaries therefore depends on real $3N$ target space momenta $(j_\nu, \mu_\nu, \bar{\mu}_\nu)_{\nu=1,\ldots,N}$ and $2g - 1$ complex moduli. These add up to $3N + 4g - 2$ real moduli. On the Liouville side, we count N real target space momenta α_ν in addition to the (complex) position of $N + 2g - 2$ Liouville degenerate fields and an overall complex pre-factor u. The total number of real parameters is therefore $3N + 4g - 2$, just as in the WZNW theory.

2. Derivation of the correspondence — Genus 0

Our first task is to explain how the correspondence emerges from a path integral 'definition' of the WZNW model. Indeed, we will be able to recover the formula of Ribault and Teschner through some formal path integral manipulations. In the second subsection we briefly review how the differential equations on both sides of the correspondence are mapped onto each other. At genus 0 this is not new, but it will help to understand the corresponding analysis at higher genus later on. Finally, we shall comment on the hidden problems of the formal path integral approach and the precise interpretation of our results.

2.1 Path integral derivation of the correspondence

Let us begin with a little bit of background on the H_3^+ WZNW model. In the most common presentation, the action involves the three fields $\gamma, \bar{\gamma}$ and ϕ corresponding to the coordinates that parametrize the 2D boundary of H_3^+ and the radial direction, respectively. We will not work with this version but pass to a first order formulation which includes two additional fields β and $\bar{\beta}$ of conformal weight $(h_\beta, 0) = (1, 0)$ and $(0, h_{\bar{\beta}}) = (0, 1)$. Throughout this work we shall work in conformal gauge, where the world-sheet metric and curvature are determined by some function ρ through

$$ds^2 \;=\; |\rho(z)|^2\, dz d\bar{z} \quad , \quad \sqrt{g}\mathcal{R} \;=\; -4\partial\bar{\partial} \ln |\rho|^2 \quad .$$

In this particular gauge, the action of the H_3^+ WZNW model takes the following form (see e.g. [14, 15])

$$S[\phi, \gamma, \beta] \;=\; \frac{1}{2\pi} \int d^2w \left(\bar{\partial}\phi\partial\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + \frac{Q_\phi}{4} \sqrt{g}\mathcal{R}\, \phi - b^2 \beta\bar{\beta} e^{2b\phi} \right) \quad . \tag{2.1}$$

Here, the parameter b is related to the level of the WZNW model through $b^{-2} = k - 2$ and the background charge Q_ϕ is given by $Q_\phi = b$. Let us note that the usual action of the

H_3^+ model emerges after integration over β and $\bar{\beta}$. The total central charge of the model is computed from the level k using

$$c(H_3^+) \;=\; \frac{3k}{k-2} \;=\; 3+6b^2 \;=\; 2+(1+6b^2) \;\; .$$

In the last step we split the central charge into the contribution $c(\beta\gamma) = 2$ from the $\beta\gamma$ system and the remainder $c(\phi) = 1+6b^2$ which originates from the bosonic field ϕ with background charge $Q_\phi = b$.

The second ingredients we shall need before we can study correlators of the H_3^+ model are the vertex operators of the theory. In the so-called μ-basis these read

$$V_j(\mu|z) \equiv |\rho(z)|^{2\Delta_j^H} \, |\mu|^{2j+2} \, e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})} e^{2b(j+1)\phi(z,\bar{z})} \; . \tag{2.2}$$

In conformal gauge, the factor involving $\rho(z)$ must be included in order for V to transform as a primary of weight zero under conformal transformations. The quantity Δ_j^H in the exponent is given by

$$\Delta_j^H \;=\; -b^2 j(j+1) \;\; .$$

One may consider μ and $\bar{\mu}$ as Euclidean light cone momenta. Similarly, the parameter j is related to the momentum in radial ϕ direction. In the μ basis, the operator products of currents with primary fields are given by

$$J^a(w)\; V_j(\mu|z) \;=\; (w-z)^{-1}\, D^a V_j(\mu|z) \quad \text{for} \quad a \;=\; \pm, 0 \tag{2.3}$$

where the generators D^a of global target space symmetries take the form

$$D^- = \mu\,, \qquad D^0 = -\mu\partial_\mu\,, \qquad D^+ = \mu\partial_\mu^2 - \frac{j(j+1)}{\mu} \tag{2.4}$$

and similarly for the remaining three generators $\bar{D}^a$. The μ basis is the most convenient one for what we are about to discuss.

Our aim is to compute the N-point function of primary fields in the H_3^+ model, i.e.

$$\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^H \;=\; \int \mathcal{D}\phi\mathcal{D}\gamma\mathcal{D}\beta \; e^{-S[\phi,\gamma,\beta]} \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu,|z_\nu) \;\; .$$

Our first step is to integrate out the fields γ and $\bar{\gamma}$. This is rather easy because they appear only linearly in both the action and the exponents of the vertex operators. Hence, the integration leads to a simple δ function constraint on the coefficients of γ and $\bar{\gamma}$, i.e.

$$\bar{\partial}\beta(w) \;=\; 2\pi\sum_{\nu=1}^{N}\mu_\nu\delta^2(w-z_\nu) \quad , \quad \partial\bar{\beta}(\bar{w}) \;=\; -2\pi\sum_{\nu=1}^{N}\bar{\mu}_\nu\delta^2(\bar{w}-z_\nu)\;. \tag{2.5}$$

The distribution δ^2 on the right hand side is normalized such that $\int d^2z\delta^2(z)f(z,\bar{z}) \;=\; f(0)$. Let us stress that a meromorphic differential β with the property (2.5) exists if and only if the sum $\sum\mu_\nu$ vanishes. Once this condition is met, the integration of $\bar{\partial}\beta$ and $\partial\bar{\beta}$ with respect to the world-sheet coordinate w is immediately performed using the simple auxiliary formulas

$$\bar{\partial}(1/z) \;=\; \partial(1/\bar{z}) \;=\; 2\pi\delta^2(z) \ .$$

The result is

$$\rho(w)\beta(w) \;=\; \sum_{\nu=1}^{N} \frac{\rho(w)\mu_\nu}{w - z_\nu} \quad . \tag{2.6}$$

A similar equation holds for $-\bar{\beta}$. We are expressing it in terms of $\rho\beta$ and $\bar{\rho}\bar{\beta}$ rather than β and $\bar{\beta}$ because the former are proper one-differentials. The crucial idea now is to re-parametrize $\rho\beta$ using simple facts about meromorphic one-differentials on the sphere. To begin with, we recall that for any one-differential, the number of poles exceeds the number of zeroes by two. Hence β must have $N-2$ zeroes whose locations on the sphere we denote by $w = y_i$. Furthermore, a differential is uniquely characterized by the position of its zeroes and poles up to an overall factor u. Consequently, we can rewrite $\rho\beta$ in the form

$$\rho(w)\beta(w) \;=\; u\, \frac{\prod_{i=1}^{N-2}(w - y_i)}{\prod_{\nu=1}^{N}(w - z_\nu)} \;=:\; u\, \rho(w)\, \mathcal{B}_0(y_i, z_\nu; w) \ . \tag{2.7}$$

Thereby, we have now replaced the N parameters μ_ν subject to constraint $\sum_\nu \mu_\nu = 0$ through $N-2$ coordinates y_i and a global factor u. We can recover the residues $\rho(z_\nu)\mu_\nu$ of $\rho\beta$ from the new variables y_i and u through

$$\rho(z_\nu)\, \mu_\nu \;=\; u\, \frac{\prod_{j=1}^{N-2}(z_\nu - y_j)}{\prod_{\mu\neq\nu}^{N}(z_\nu - z_\mu)} \ . \tag{2.8}$$

The new variables may be used to rewrite the $\boldsymbol{\delta}$ function resulting from the integration over γ and $\bar{\gamma}$,

$$\boldsymbol{\delta}^2(\bar{\partial}\beta(w) - 2\pi \sum_{\nu=1}^{N} \mu_\nu \delta^2(w - z_\nu)) \;=\; \delta^2(\sum_{\nu=1}^{N} \mu_\nu)\ \boldsymbol{\delta}^2\left(\beta - u\mathcal{B}_0(y_i, z_\mu, w)\right) \ . \tag{2.9}$$

The Jacobian for the transformation from $\bar{\partial}\beta, \partial\bar{\beta}$ to $\beta, \bar{\beta}$ is trivial on the sphere. Once eq. (2.9) has been inserted into our path integral, we perform the integral over the fields β and $\bar{\beta}$ to obtain

$$\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^H = |u|^2\, \delta^2(\sum_{\nu=1}^{N} \mu_\nu) \int \mathcal{D}\phi e^{-\frac{1}{2\pi}\int d^2w\ \left(\bar{\partial}\phi\partial\phi + \frac{Q_\phi}{4}\sqrt{g}\mathcal{R}\phi + b^2|\mathcal{B}_0|^2\, e^{2b\phi}\right)} \times$$
$$\times \prod_{\nu=1}^{N} |\rho(z_\nu)|^{2\Delta_\nu^H}\, |u|^{-2(j_\nu+1)} |\mu_\nu|^{2(j_\nu+1)} e^{2b(j_\nu+1)\phi(z_\nu)} \ .$$

In writing this formula we have also shifted the zero mode of the bosonic field ϕ by $\phi \mapsto \phi - (1/b)\ln|u|$. This removes the u dependence from the interaction term but introduces an additional factor $|u|^2$ through the coupling of ϕ to the world-sheet curvature.

In order to prepare for the second step of our calculation we observe that the exponential field $\exp(2b\phi)$ always comes multiplied with $|\mathcal{B}_0|^2$. For the corresponding term in the action, this has been spelled out explicitly. In the case of the vertex operators, the prefactor is hidden in $|\mu_\nu|^2$ once we insert eq. (2.8). This suggests to introduce a new bosonic field φ through

$$\varphi(w,\bar{w}) \;:=\; \phi(w,\bar{w}) + \frac{1}{2b}\left(\sum_i \ln|w-y_i|^2 - \sum_\nu \ln|w-z_\nu|^2 - \ln|\rho(w)|^2\right) \tag{2.10}$$

where the term in brackets is $\ln|\mathcal{B}_0|^2$. Acting with $\partial\bar{\partial}$ gives

$$\partial\bar{\partial}\varphi(w,\bar{w}) \;=\; \partial\bar{\partial}\phi(w,\bar{w}) + \frac{\pi}{b}\left(\sum_i \delta^2(w-y_i) - \sum_\nu \delta^2(w-z_\nu)\right) - \frac{1}{2b}\partial\bar{\partial}\ln|\rho(w)|^2 \quad .$$

Before we insert our formulas for φ we have to comment on one small subtlety. In fact, at various places we will have to evaluate the quantity $\ln|w-z_\nu|$ at the point $w=z_\nu$. This leads to well-known singularities which need to be regularized. Following Polyakov [16], we shall use the prescription

$$\left(\lim_{w\to z}\ln|w-z|^2\right)_{\text{fin}} \;:=\; -\ln|\rho(z)|^2 \quad . \tag{2.11}$$

The right hand side is finite and has the same behavior under conformal transformations as the divergent left hand side. Using our change of variables from ϕ to φ along with the rule (2.11) it is easy to see that

$$|\mu_\nu|^{2(j_\nu+1)}e^{2b(j_\nu+1)\phi(z_\nu)} \;=\; |u|^{2(j_\nu+1)}\;|\rho(z_\nu)|^{-2(j_\nu+1)}\;e^{2b(j_\nu+1)\varphi(z_\nu)} \quad . \tag{2.12}$$

After these comments we are prepared to perform our substitution from ϕ to φ. We do not want to spell out all the details, but let us observe that

$$-\frac{1}{2\pi}\int d^2w\,\bar{\partial}\phi\partial\phi \;=\; -\frac{1}{2\pi}\int d^2w\,\left(\bar{\partial}\varphi\partial\varphi + \frac{1}{4b}\sqrt{g}\mathcal{R}\varphi\right) - \frac{1}{b}\left(\sum_{i=1}^{N-2}\varphi(y_i) - \sum_{\nu=1}^{N}\varphi(z_\nu)\right) + \ldots$$

where the ... stand for terms that do not contain the field φ. There are a few things we can read off from this result. To begin with, the background charge Q_ϕ of the bosonic field ϕ gets shifted by an amount $\Delta Q = 1/b$ to the new background charge $Q_\varphi = b + 1/b$. Furthermore, the exponents of the vertex operator insertions at z_ν are all shifted by $\varphi(z_\nu)/b$. The precise relation is

$$e^{2b(j_\nu+1)\varphi(z_\nu)}\;e^{\frac{1}{b}\varphi(z_\nu)} \;=\; |\rho(z_\nu)|^{2(j_\nu+1)}\;e^{2b(j_\nu+1+\frac{1}{2b^2})\varphi(z_\nu)} \quad .$$

Note that the additional power of ρ is required to match the behavior of both sides under conformal transformations. More strikingly, new vertex operators $\exp(-\varphi(w)/b)$ are inserted at the $N-2$ points y_i. With a little bit of additional care we can also work out the additional φ-independent factors. The result is

$$\left\langle\,\prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu)\,\right\rangle^H = |\Theta_N(u,y_i,z_\nu)|^2\,\delta^2(\sum_{\nu=1}^{N}\mu_\nu)\int\mathcal{D}\varphi\,e^{-\frac{1}{2\pi}\int d^2w\,\left(\bar{\partial}\varphi\partial\varphi + \frac{Q_\varphi}{4}\sqrt{g}\mathcal{R}\varphi + b^2e^{2b\varphi}\right)}$$
$$\times\;\prod_{\nu=1}^{N}|\rho(z_\nu)|^{2\Delta_\nu^L}\,e^{2\left(b(j_\nu+1)+\frac{1}{2b}\right)\varphi(z_\nu)}\;\prod_{i=1}^{N-2}|\rho(y_i)|^{2\Delta^L_{-1/2b}}\,e^{-\frac{1}{b}\varphi(y_i)}$$

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where, following [7], we collected various factors in the function Θ_N defined by

$$\Theta_N(u, y_i, z_\nu) \;=\; u \prod_{\mu<\nu}^{N} (z_\mu - z_\nu)^{\frac{1}{2b^2}} \prod_{i<j}^{N-2} (y_i - y_j)^{\frac{1}{2b^2}} \prod_{\mu=1}^{N} \prod_{i=1}^{N-2} (z_\mu - y_i)^{-\frac{1}{2b^2}} , \tag{2.13}$$

and we introduced the quantities

$$\Delta^L_\alpha \;=\; \alpha(Q_\varphi - \alpha) \quad \text{for} \quad \alpha \;=\; -1/2b \quad \text{or} \quad \alpha \;=\; \alpha_\nu \;=\; b(j_\nu + 1) + \frac{1}{2b} \quad . \tag{2.14}$$

Except for the first two factors, the result may now be re-expressed in terms of correlation functions of Liouville theory,

$$\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^H \;=\; \delta^2(\sum_{\nu=1}^{N} \mu_\nu) |\Theta_N(u, y_i, z_\nu)|^2 \left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu) \prod_{i=1}^{N-2} V_{-1/2b}(y_i) \right\rangle^L . \tag{2.15}$$

The Liouville correlator on the right hand side is evaluated with a bulk cosmological constant $\mu_B = 4b^2$ and the primaries are defined by

$$V_\alpha(z) \;=\; |\rho(z)|^{2\Delta^L_\alpha} e^{2\alpha\varphi} \quad . \tag{2.16}$$

Up to an overall constant, our result coincides with the formula found by Ribault and Teschner in [7].

2.2 From KZ to BPZ differential equations

The correlation functions on both sides of the correspondence are known to satisfy certain differential equations. Throughout this subsection we shall set $\rho = 1$. Liouville correlators are known to obey the Belavin-Polyakov-Zamolodchikov (BPZ) second order differential equations that come with the $N-2$ insertions of the degenerate field $V_{-1/2b}$. In fact, this field is well known to possess a singular vector on the second level, or, equivalently, to satisfy the differential equation

$$\partial_y^2 V_{-1/2b}(y) + b^{-2} : T(y) V_{-1/2b}(y) : \;=\; 0 \quad .$$

Such an equation holds for each of the $N-2$ degenerate fields at the points y_i. Using the Ward identities for the Virasoro field $T(y_i)$, we can convert these singular vector equations into $N-2$ second order differential equations for the Liouville correlators Ω^L ,

$$D^{\mathrm{L}}_i\, \Omega^L(z_\nu, y_i) \;=\; 0 \qquad \text{where} \quad \Omega^L(z_\nu, y_i) \;=\; \left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu) \prod_{i=1}^{N-2} V_{-1/2b}(y_i) \right\rangle^L \tag{2.17}$$

and the differential operators D^{L}_i were first found by Belavin, Polyakov and Zamolodchikov to be of the form

$$D^{\mathrm{L}}_i \;=\; b^2 \frac{\partial^2}{\partial y_i^2} + \sum_{\nu=1}^{N} \left(\frac{\Delta^L_\nu}{(y_i - z_\nu)^2} + \frac{1}{y_i - z_\nu} \partial_\nu \right) + \sum_{j \neq i}^{N} \left(\frac{\Delta^L_{-1/2b}}{(y_i - y_j)^2} + \frac{1}{y_i - y_j} \partial_j \right) . \tag{2.18}$$

Here, $\partial_i = \partial/\partial y_i$ and $\partial_\nu = \partial/\partial z_\nu$ and the conformal dimensions are given by $\Delta^L_\nu = \alpha_\nu(Q - \alpha_\nu)$, as before.

Let us now turn to the WZNW model. Its correlators are well known to satisfy the Knizhnik-Zamolodchikov (KZ) differential equations. These emerge from insertions of the Sugawara singular vector

$$T(w) + b^2 \left(: J^0(w)J^0(w) : -\frac{1}{2}(: J^-(w)J^+(w) : + : J^+(w)J^-(w) :) \right) \; = \; 0$$

into WZNW correlation functions. Since we can insert this at any point on the sphere, we certainly get an infinite number of equations. But these are not all independent. In fact, evaluation of the residues of the first order poles at $w = z_\nu$ gives N independent first order equations. Hence, there are two more equations on the WZNW side than on the Liouville side. Closer inspection shows that the $N-2$ BPZ differential equations correspond inserting the Sugawara tensor at the $N-2$ points $w = y_i$ on the world-sheet of the WZNW model. At these special points, the differential equations from the Sugawara singular vector simplify considerably. To see this we need the operator product expansions between the currents $J^\pm$ and J^0 and the vertex operators in the μ basis (2.3). From the current Ward identities and the very definition of the points y_i we conclude that

$$J^-(y_i) \; = \; \sum_{\nu=1}^{N} \frac{\mu_\nu}{y_i - z_\nu} \; = \; 0 \quad .$$

By similar reasoning, the current $J^0(y_i)$ is easily seen to simplify as follows

$$J^0(y_i) \; = \; -\sum_{\nu=1}^{N} \frac{\mu_\nu}{y_i - z_\nu} \frac{\partial}{\partial \mu_\nu} \; = -\frac{\partial}{\partial y_i}$$

where the second equality follows from the explicit chance of variables relating μ_ν with y_i and u. Once we insert all these results into the Sugawara singular vector, we obtain the following $N-2$ differential equations

$$D^{\mathrm{H}}_i \Omega^H(z_\nu, \mu_\nu) \; = \; 0 \qquad \text{where} \quad \Omega^H(z_\nu, \mu_\nu) \; = \; \left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^H \quad . \tag{2.19}$$

Using the relation (2.8), the differential operators D^H_i for H^+_3 can be shown to acquire the following form

$$D^{\mathrm{H}}_i \; = \; b^2 \frac{\partial}{\partial y_i^2} + \sum_{\nu=1}^{N} \left(\frac{1}{y_i - z_\nu} \partial_\nu + \frac{\Delta^H_\nu}{(y_i - z_\nu)^2} \right) , \tag{2.20}$$

where $\Delta^H_\nu = -b^2 j_\nu(j_\nu + 1)$ are the conformal dimensions of our WZNW primary fields, as before. Note that D^{H}_i does not depend on any of the y_j with $j \neq i$. In this sense, the transformation (2.8) leads to a *separation of variables* [17]. Let us stress that the derivatives ∂_ν in eq. (2.19) are to be taken with fixed μ_ν whereas the variables y_i are kept fixed when evaluating ∂_ν in the BPZ differential equations above. Whenever it is relevant, we shall explicitly distinguish between the two derivatives ∂_ν,

$$\partial_\nu^{\rm H} \;=\; \left(\frac{\partial}{\partial z_\nu}\right)_\mu \quad , \quad \partial_\nu^{\rm L} \;=\; \left(\frac{\partial}{\partial z_\nu}\right)_y \quad .$$

It is certainly possible to express e.g. all the derivatives $\partial_\nu^{\rm H}$ in terms of $\partial_\nu^{\rm L}, \partial_i$ and $\partial/\partial u$. We shall only need the special combination

$$\delta_i \;:=\; \sum_\nu \frac{1}{y_i - z_\nu}\,\partial_\nu^{\rm H} \;=\; \sum_\nu \frac{1}{y_i - z_\nu}\left(\partial_\nu^{\rm L} + \partial_i\right) - \sum_{j\neq i} \frac{1}{y_i - y_j}\left(\partial_i - \partial_j\right) \quad . \tag{2.21}$$

With this auxiliary formula it is then straightforward to check that

$$\left(\Theta_N^{-1} D_i^{\rm H} \Theta_N - D_i^{\rm L}\right)\,\Omega^L(z_\nu, y_i) \;=\; 0\;. \tag{2.22}$$

The correspondence between differential equation was an important ingredient in [7] for proving the relation (2.15). We have derived the latter within the path integral approach and therefore the differential equations are guaranteed to be mapped onto each other.

2.3 Comments on the path integral derivation

It is well known that the path integral definition of both the WZNW model and of Liouville theory has some issues that are related to the non-compactness of the background (see e.g. [18] for a review). If one splits all fields into their zero modes and fluctuations, one can integrate out the zero modes. The procedure results in expression for the correlators which are either divergent or hard to give an exact definition. In the case of Liouville theory, for instance, one obtains

$$\langle \prod_{\nu=1}^N e^{2\alpha_\nu \varphi(z_\nu,\bar z_\nu)} \rangle^L \;=\; \int \mathcal{D}\tilde\varphi\, e^{-S_{\rm LD}[\tilde\varphi]} \prod_{\nu=1}^N e^{2\alpha_\nu \tilde\varphi(z_\nu,\bar z_\nu)} \frac{\Gamma(-s)}{2b}\left(\mu_B \int d^2 w e^{2b\tilde\varphi}\right)^s$$

where $sb = Q_\varphi - \sum_{\nu=1}^N \alpha_\nu$. In this formula, the integration over the fluctuation field $\tilde\varphi$ is weighted with the measure of the free linear dilaton theory, but the integrand contains additional insertions of screening charges. The latter are raised to some possibly non-integer power s and they are multiplied with coefficients that diverge for positive integer powers. These features have a simple explanation. In our non-compact target space, the correlation functions are expected to possess poles in momentum space which come from the integration over the infinite region of the target space where the interaction is negligible. The path integral computation we have just sketched detects (some of) these divergencies and can be turned into a rigorous computation of the associated residues.

If we are not willing to give the path integral any more credit, then our results above only imply that

$$\left\langle \prod_{\nu=1}^N V_{j_\nu}(\mu_\nu|z_\nu)\; S_\phi^s \right\rangle_{LD^\phi}^{\beta\gamma} \;=\; \delta^2(\sum_{\nu=1}^N \mu_\nu)|\Theta_N(u, y_i, z_\nu)|^2 \times \tag{2.23}$$

$$\times \left\langle \prod_{\nu=1}^N V_{\alpha_\nu}(z_\nu) \prod_{i=1}^{N-2} V_{-1/2b}(y_i)\; S_\varphi^s \right\rangle_{LD^\varphi} \quad .$$

Here, S^s_ϕ and S^s_φ denote the screening charges of the WZNW model and Liouville field theory, respectively. They are given by the following expressions

$$S^s_\phi \;=\; -\int d^2w\ \beta(w)\bar\beta(\bar w)\, e^{2b\phi(w,\bar w)} \quad , \quad S^s_\varphi \;=\; \int d^2w\ e^{2b\varphi(w,\bar w)}\ . \tag{2.24}$$

The correlation functions on both sides of the equality are to be computed in the free linear dilaton theory. On the left hand side, we use the $\beta\gamma$ system with central charge $c = 2$ and a linear dilaton with background charge $Q_\phi = b$. On the right hand side, the correlator is to be computed for a linear dilaton background with $Q_\varphi = b + 1/b$.

Correlation functions in the WZNW model and in Liouville field theory are well known to possess a second series of poles that are not explained by insertions of the screening charges (2.24). The residues of these poles can still be computed from free field theory with the help of so-called dual screening charges. For our two models these read

$$S^d_\phi \;=\; -\int d^2w\ \beta(w)^{\frac{1}{b^2}}\bar\beta(\bar w)^{\frac{1}{b^2}}\, e^{\frac{2}{b}\phi(w,\bar w)} \quad , \quad S^d_\varphi \;=\; \int d^2w\ e^{\frac{2}{b}\varphi(w,\bar w)}\ . \tag{2.25}$$

Note that the exponents $1/b^2 = k-2$ can be integer for integer level $k \geq 2$. We have indeed checked by explicit computation that equation (2.23) remains valid if the screening charges S^s_ϕ and S^s_φ are replaced by the dual ones. Similar calculations for the H_3^+ model involving both screening charges S^s_ϕ and S^d_ϕ can be found, e.g., in [19] (see also references therein for earlier works on this subject). We shall perform free field computations for correlators on surfaces of genus g at the end of section 4.

3. Generalizing the correspondence to the torus

Motivated by our rather simple path integral derivation, we would now like to extend the H_3^+-Liouville relation beyond tree level to higher genus correlators. We shall study the general case in the next section and restrict ourselves to the torus for now since many of the formulas can be made very explicit at $g = 1$.

3.1 The H_3^+ model on the torus

We start from the H_3^+ WZNW model with level k and compute N-point functions on a torus with moduli parameter τ using the first order formulation in terms of ϕ, β, γ. As explained in the introduction, we would like to introduce a bit more freedom by admitting some non-trivial boundary conditions for ϕ and the $\beta\gamma$ system. To be precise, we assume the fields ϕ, γ and β to satisfy

$$\beta(w+m+n\tau) = \ e^{2\pi i n\lambda}\beta(w)\,, \qquad\qquad \gamma(w+m+n\tau) \;=\; e^{-2\pi i n\lambda}\gamma(w)\,, \tag{3.1}$$

$$\phi(w+m+n\tau, \bar w+m+n\bar\tau) \;=\; \phi(w) + \frac{2\pi n \mathrm{Im}\lambda}{b} \tag{3.2}$$

with $n, m \in \mathbb{Z}$ and λ some complex parameter. Once such twisted boundary conditions have been introduced for the field β, the conditions on γ and ϕ follow if we require the action to be single valued. Of course, we assume similar twisted boundary conditions with twist parameter $\bar\lambda$ to hold for the anti-holomorphic components of the $\beta\gamma$ system.

There is one term in the action, namely the coupling of the field ϕ to the world-sheet curvature, that requires a bit of additional care. Since our field ϕ is multivalued, the term $\sqrt{g}\mathcal{R}\,\phi$ cannot possibly give the right prescription. Instead, we must decompose ϕ into a twisted zero mode part $\phi_{\rm sol}$ (the index 'sol' stands for *solitonic*) and a doubly periodic fluctuation ϕ_q , i.e.

$$\phi \;=\; \phi_q + \phi_{\rm sol} \qquad \text{where} \qquad \phi_{\rm sol}(w,\bar w) \;=\; \frac{2\pi}{\tau_2 b}{\rm Im}\lambda{\rm Im}w \;, \tag{3.3}$$

where, by construction, ϕ_q now satisfies $\phi_q(w+m+n\tau,\bar w+m+n\bar\tau) \;=\; \phi_q(w)$. The linear dilaton term in the action couples the world-sheet curvature to the single valued fluctuation field ϕ_q rather than to ϕ itself,

$$S_{\rm LD}[\phi] \;=\; \frac{Q_\phi}{8\pi}\int d^2w\,\sqrt{g}\mathcal{R}\,\phi_q \quad . \tag{3.4}$$

All other terms in the action (2.1) remain the same as before. Similarly, the expression for vertex operators (2.2) does not require any modification.

To a large extend the path integral computation of the N-point correlation function follows the same steps as before. The quantities we would like to compute are given by

$$\left\langle \prod_{\nu=1}^N V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^H_{(\lambda,\tau)} \;=\; \int \mathcal{D}^\lambda\phi\mathcal{D}^\lambda\gamma\mathcal{D}^\lambda\beta\; e^{-S[\phi,\gamma,\beta]}\prod_{\nu=1}^N V_{j_\nu}(\mu_\nu,|z_\nu)\,,$$

where the integration is to be performed over all field configurations on a torus with modulus τ satisfying the boundary conditions (3.1) and (3.2) stated above. Note that our correlation function is not yet normalized by dividing the partition function Z^H. We shall further comment on this below. After integration over γ and $\bar\gamma$ we obtain the same condition (2.5) for the derivatives of β and $\bar\beta$ on the sphere. But this time it has different consequences for β and $\bar\beta$. In fact, we need a bit of preparation before we are able to spell out the analogue of the important equation (2.7).

On the torus, the integration of equation (2.5) will lead to a new function that can be constructed out of Jacobi's theta function

$$\theta(z) = \;-\sum_{n\in\mathbb{Z}} e^{i\pi(n+\frac12)^2\tau+2\pi i(n+\frac12)(z+\frac12)}\,, \qquad \theta(z+m+n\tau) = -e^{-i\pi n(2z+\tau)}\theta(z)\;. \tag{3.5}$$

Out of θ we can build a new function σ_λ with a simple pole and the same periodicity properties that we required for β,

$$\sigma_\lambda(z,w) \;=\; \frac{\theta(\lambda-(z-w))\theta'(0)}{\theta(z-w)\theta(\lambda)} \quad . \tag{3.6}$$

Indeed, from the shift properties of Jacobi's theta function θ it is easy to derive the following property of σ_λ,

$$\sigma_\lambda(z+m+n\tau,w) \;=\; e^{2\pi in\lambda}\sigma_\lambda(z,w)\;. \tag{3.7}$$

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Now let us return to the integration of the two equations (2.5). The right hand side tells us that the twisted meromorphic differential $\rho(w)\beta(w)$ possesses N poles in the positions $w = z_\nu$ with residues $\rho(z_\nu)\mu_\nu$. The solution to these conditions is unique, as long as the twist parameter λ does not vanish. It can be written down in terms of the σ_λ as

$$\rho(w)\beta(w) \;=\; \sum_{\nu=1}^{N} \rho(w)\mu_\nu\sigma_\lambda(w,z_\nu) \;=\; u\frac{\prod_{i=1}^{N}\theta(w-y_i)}{\prod_{\nu=1}^{N}\theta(w-z_\nu)} \;=:\; u\,\rho(w)\,\mathcal{B}_1(y_i,z_\nu;w)\ . \tag{3.8}$$

The second equality is a torus version of Sklyanin's separation of variables and it requires a few comments. On a torus, meromorphic one-differentials possess the same number N of poles and zeroes. Moreover, the positions $w = z_\nu$ and $w = y_i, i = 1,\dots,N$, of both zeroes and poles determine the differential up to an overall factor u. For our correspondence between the WZNW model and Liouville theory, it is again crucial to parametrize $\rho\beta$ through u and y_i rather than λ and μ_ν. The relation between the two sets of parameters can be worked out easily (see also [20, 21])

$$\rho(z_\nu)\mu_\nu \;=\; u\frac{\prod_{i=1}^{N}\theta(z_\nu-y_i)}{\theta'(0)\prod_{\mu\neq\nu,\mu=1}^{N}\theta(z_\nu-z_\mu)} \quad , \quad \lambda \;=\; \sum_{i=1}^{N} y_i - \sum_{\nu=1}^{N} z_\nu\ . \tag{3.9}$$

What we have shown so far is that the integration over the fields γ and $\bar\gamma$ in the WZNW model leads to the following $\boldsymbol{\delta}$ function

$$\boldsymbol{\delta}^2(\bar\partial\beta(w) - 2\pi\sum_{\nu=1}^{N}\mu_\nu\delta^2(w-z_\nu)) \;=\; |\mathrm{det}\partial|_\lambda^{-2}\boldsymbol{\delta}^2(\beta(w) - u\,\mathcal{B}_1(y_i,z_\nu;w))\ . \tag{3.10}$$

This replaces the related formula (2.9) in the genus zero analysis. The factor $|\det\partial|_\lambda^{-2}$ is the Jacobian that arises when we change from $\delta^2(\partial\beta\cdots)$ to $\delta^2(\beta(w)\cdots)$. We have placed a subscript λ in the Jacobian to remind us that ∂ is considered as an operator on twisted one-differentials. Let us now perform the integration over β and $\bar\beta$ to obtain

$$\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^{H}_{(\lambda,\tau)} = \frac{1}{|\det\partial|_\lambda^2}\int \mathcal{D}^\lambda\phi e^{-\frac{1}{2\pi}\int d^2w\ \left(\bar\partial\phi\partial\phi + \frac{Q_\phi}{4}\sqrt{g}\mathcal{R}\phi + b^2|\mathcal{B}_1|^2\,e^{2b\phi}\right)} \times$$
$$\times \prod_{\nu=1}^{N} |\rho(z_\nu)|^{2\Delta_\nu^J}\,|u|^{-2(j_\nu+1)}|\mu_\nu|^{2(j_\nu+1)}e^{2b(j_\nu+1)\phi(z_\nu)}\ .$$

As in our genus zero analysis we have shifted the zero mode of the field ϕ to remove the $|u|^2$ from the interaction term. Because the Euler characteristics of the torus vanishes, the path integral is multiplied with a factor $|u|^0 = 1$.

We have now reached the point at which we change variables for the remaining integration over ϕ such that the highly non-trivial factor $|\mathcal{B}_1|^2$ also gets removed from the interaction. In complete analogy to the genus zero case we introduce

$$\varphi(w,\bar w) \;:=\; \phi(w,\bar w) + \frac{1}{2b}\left(\sum_i \ln|\theta(w,y_i)|^2 - \sum_\nu \ln|\theta(w,z_\nu)|^2 - \ln|\rho(w)|^2\right)\ . \tag{3.11}$$

By construction, the new field φ is periodic under shift by $n+m\tau$, even though the original field ϕ and the θ functions are not. It will be advantageous to replace $|\theta|^2$ functions by some function F,

$$F(z,w) \ := \ e^{-\frac{2\pi}{\tau_2}(\mathrm{Im}(z-w))^2} \left|\frac{\theta(z-w)}{\theta'(0)}\right|^2 , \tag{3.12}$$

which is easily seen to be invariant under shifts by integers and integer multiples of the modulus τ,

$$F(z+m+n\tau,w) \ = \ F(z,w) \quad .$$

The new function F allows us to rewrite the relation between φ and ϕ in terms of the single valued fluctuation field $\phi_q = \phi - \phi_{\mathrm{sol}}$ that we introduced in eq. (3.3),

$$\varphi(w,\bar{w}) := \phi_q(w,\bar{w}) + \frac{1}{2b}\left(\sum_i \ln F(w,y_i) - \sum_\nu \ln F(w,z_\nu) - \ln|\rho(w)|^2 + S_1\right) \tag{3.13}$$

with $S_1 = \frac{2\pi}{\tau_2}(\sum_i y_i^2 - \sum_\nu z_\nu^2)$. Let us also observe that the decomposition of the field ϕ into ϕ_{sol} and ϕ_q is such that their contributions to the kinetic term decouple,

$$-\frac{1}{2\pi}\int d^2w\partial\phi\bar{\partial}\phi = -\frac{1}{2\pi}\int d^2w\partial\phi_{\mathrm{sol}}\bar{\partial}\phi_{\mathrm{sol}} - \frac{1}{2\pi}\int d^2w\partial\phi_q\bar{\partial}\phi_q \tag{3.14}$$

$$\text{with} \quad \frac{1}{2\pi}\int d^2w\partial\phi_{\mathrm{sol}}\bar{\partial}\phi_{\mathrm{sol}} \ = \ \frac{\pi(\mathrm{Im}\lambda)^2}{b^2\tau_2} \quad . \tag{3.15}$$

Now we can proceed exactly as before, with the function F replacing $|z-w|^2$. In particular, the properties of F can be used to evaluate $\partial\bar{\partial}\varphi$ as

$$\partial\bar{\partial}\varphi(w,\bar{w}) := \partial\bar{\partial}\phi_q(w,\bar{w}) + \frac{\pi}{b}\left(\sum_i \delta^2(w-y_i) - \sum_\nu \delta^2(w-z_\nu)\right) - \frac{1}{2b}\partial\bar{\partial}\ln|\rho(w)|^2, \tag{3.16}$$

a result that agrees exactly with the corresponding formula at genus zero. The outcome of a short and straightforward computation is

$$\begin{aligned}\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu)\right\rangle^{H}_{(\lambda,\tau)} &= e^{-\frac{\pi(\mathrm{Im}\lambda)^2}{b^2\tau_2}} \frac{|\Theta_N^{g=1}(y_i,z_\nu,\tau)|^2}{|\det\partial|^2_\lambda}\int \mathcal{D}\varphi\, e^{-\frac{1}{2\pi}\int d^2w\left(\bar{\partial}\varphi\partial\varphi + \frac{Q_\varphi}{4}\sqrt{g}\mathcal{R}\varphi + b^2e^{2b\varphi}\right)} \\ &\times \prod_{\nu=1}^{N}|\rho(z_\nu)|^{2\Delta_\nu^L} e^{2\left(b(j_\nu+1)+\frac{1}{2b}\right)\varphi(z_\nu)} \prod_{i=1}^{N}|\rho(y_i)|^{2\Delta^L_{-1/2b}} e^{-\frac{1}{b}\varphi(y_i)}\end{aligned}$$

where the pre-factor Θ_N is given by

$$|\Theta_N^{g=1}(y_i,z_\nu,\tau)|^2 \ = \ \prod_{\mu<\nu}^{N} F(z_\mu,z_\nu)^{\frac{1}{2b^2}} \prod_{i<j}^{N} F(y_i,y_j)^{\frac{1}{2b^2}} \prod_{\mu,i=1}^{N} F(z_\mu,y_i)^{-\frac{1}{2b^2}} \ . \tag{3.17}$$

So far, the correlators on both sides of the equations are not normalized. But using the following formulas (see [22] and our discussion before eq. (4.13) later on) for the genus one partition functions Z^H and Z^L of the WZNW model and Liouville theory, respectively,

$$Z^H \ = \ \frac{e^{-\frac{\pi(\mathrm{Im}\lambda)^2}{b^2\tau_2}}}{\sqrt{\tau_2}|\theta(\lambda)|^2} \quad , \qquad Z^L \ = \ \frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2} \ , \tag{3.18}$$

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along with the formula $\det \partial_\lambda = \theta(\lambda)/\eta(q)$ for the determinant of ∂ on λ-twisted differentials, we can recast the relation between correlation functions in both theories in a particularly simple form

$$\frac{1}{Z^H}\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu)\right\rangle^{H}_{(\lambda,\tau)} = |\Theta_N^{g=1}(y_i,z_\nu,\tau)|^2\,\frac{1}{Z^L}\left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu)\prod_{i=1}^{N} V_{-1/2b}(y_i)\right\rangle^{L} . \tag{3.19}$$

Our genus one relation is similar to the tree level result. Once more we managed to compute all correlators of WZNW primaries in terms of Liouville theory with the same relation between the level $k = b^{-2}+2$ and the background charge $Q_\varphi = b+1/b$ and the same bulk cosmological constant $\mu_B = 4b^2$. For each primary field V_j in the WZNW model there appears one Liouville vertex operator V_α where $\alpha = b(j+1)+1/2b$, as on the sphere. Additionally we have to insert degenerate Liouville fields, but now we need two more than on the sphere, i.e. there are N degenerate fields inserted. Their positions are determined through the light cone momenta μ_ν of the WZNW vertex operators and the twist parameter λ.

The attentive reader might be a bit surprised not to see any factor implementing the conservation of μ momentum, as on the sphere. Its absence is directly related to the fact that the correspondence has been worked out for non-zero twist parameter λ. One would certainly expect to recover μ momentum conservation in the limit $\lambda \to 0$. In order to see how this works, let us go back to relation (3.8) and insert the expansion

$$\sigma_\lambda(z,w) = \frac{1}{\lambda} + \frac{\theta'(w-z)}{\theta(w-z)} + \dots . \tag{3.20}$$

This leads to the formula

$$\beta(w) = \sum_\nu \frac{\mu_\nu}{\lambda} + \sum_\nu \mu_\nu\,\partial_w \ln\theta(w-z_\nu) + \dots .$$

For β to stay finite in the limit $\lambda \to 0$, we need to assume that the total μ-momentum $\sum_\nu \mu_\nu$ tends to zero when we send $\lambda \to 0$. In fact, if $\sum_\nu \mu_\nu$ vanishes fast enough, we obtain well defined expressions at $\lambda = 0$.

3.2 Relation between differential equations on torus

Having established a simple relation between correlation functions of the WZNW model and Liouville field theory on the torus it seems worthwhile to look once more at the differential equations that determine correlators in both models and to check that our relation (3.19) correctly intertwines between them.

Let us start on the side of Liouville field theory. The vertex operator $V_{-1/2b}$ belongs to a degenerate representation with a null vector $(b^2(L_{-1})^2 + L_{-2})|-1/2b\rangle$ on the second level. Since we have N such degenerate fields in our Liouville correlation function, we obtain N second order differential equations,

$$D_i^{\mathrm{L}}(\Omega^L)(z_\nu,y_j,\tau) = 0\,, \qquad \Omega^L(z_\nu,y_i,\tau) = \left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu)\prod_{j=1}^{N} V_{-1/2b}(y_j)\right\rangle^{L} . \tag{3.21}$$

The differential operators D_i^{L} for surfaces of higher genus were worked out by Eguchi and Ooguri [12],

$$D_i^{\mathrm{L}} = b^2\frac{\partial^2}{\partial y_i^2} + \Delta^L_{-\frac{1}{2b}} 2\eta_1 + \sum_{j\neq i}\left(\xi(y_i,y_j)\frac{\partial}{\partial y_j} - \Delta^L_{-\frac{1}{2b}}\partial_i\xi(y_i,y_j)\right) +$$
$$+\sum_\nu\left(\xi(y_i,z_\nu)\frac{\partial}{\partial z_\nu} - \Delta^L_{\alpha_\nu}\partial_i\xi(y_i,z_\nu)\right) + 2\pi i\frac{\partial}{\partial\tau}\ , \tag{3.22}$$

where the special function ξ and the constant η_1 are constructed from Jacobi's θ function through

$$\xi(z,w) = \frac{\theta'(z-w)}{\theta(z-w)}\,, \qquad \eta_1 = -\frac{1}{6}\frac{\theta'''(0)}{\theta'(0)}\ . \tag{3.23}$$

Let us now address the differential equations obeyed by the correlation functions of the WZNW model on the torus, which were first worked out by Bernard and are known as Knizhnik-Zamolodchikov-Bernard (KZB) equations. These equations are obtained by inserting the Sugawara singular vector $T(w) - b^2 : J^a J_a : (w) = 0$ into the WZNW N-point correlation function Ω^H (see eq. (3.28) below for the definition of Ω^H). Ward identities for currents and the Virasoro field then give [13, 23]

$$\left[b^2 S(w) + \sum_{\nu=1}^N\left(\xi(y_a,z_\nu)\frac{\partial}{\partial z_\nu} - \Delta^H_{j_\nu}\partial_i\xi(y_i,z_\nu)\right) + 2\pi i\frac{\partial}{\partial\tau}\right]\Pi\Omega^H = 0\ . \tag{3.24}$$

Here, we introduced $\Pi = |\theta(\lambda)|^2$ and

$$S(w) = \left(\frac{\partial}{\partial\lambda} - \tilde{J}^0(w)\right)^2 - \frac{1}{2}\left(\tilde{J}^-(w)\tilde{J}^+(w) + \tilde{J}^+(w)\tilde{J}^-(w)\right)\ ,$$
$$\tilde{J}^\mp(w) = \sum_{\nu=1}^N \sigma_{\pm\lambda}(w,z_\nu)\,D_\nu^\mp \quad , \qquad \tilde{J}^0(w) = \sum_{\nu=1}^N \xi(w,z_\nu)\,D_\nu^0\ , \tag{3.25}$$

and the differential operators $D_\nu^\pm$ and D_ν^0 are the same as in eq. (2.4). Let us briefly recall the reason why the KZB equations contain a derivative with respect to the twist parameter λ. These terms arise from the Ward identities of currents. In fact, it has already been observed by Eguchi and Ooguri in [12] that the insertion of the zero modes of currents into correlators cannot be converted into differential operators acting on the usual untwisted correlation functions. It was Bernard's idea to fix this problem by introducing a dependence of conformal blocks on additional parameters. On the torus, he suggested to insert a group element g into the trace. This has the effect of twisting the boundary conditions for currents under shifts by multiples of τ. Our boundary conditions correspond to the special choice $g = \exp(-2\pi i\lambda J_0^0)$. Actually, it had been observed by Bernard already that a single twist parameter λ suffices on the torus.

As in subsection 2.2, our strategy now is to evaluate the KZB equations at the N special points y_i and then to compare the result with the N differential equations for the Liouville correlator. From the relation (3.9) between μ_i, λ and y_i, u one may derive

$$\frac{\partial}{\partial y_i} = \frac{\partial}{\partial\lambda} + \sum_{\nu=1}^N \xi(y_i,z_\nu)\,\mu_\nu\frac{\partial}{\partial\mu_\nu}\,, \qquad u\frac{\partial}{\partial u} = \sum_{\nu=1}^N \mu_\nu\frac{\partial}{\partial\mu_\nu}\ . \tag{3.26}$$

These relations between derivatives can be inserted into our formulas for the differential operators $\tilde{J}^{\pm}(w)$ and $\tilde{J}^0(w)$, evaluated at the points $w = y_i$, to obtain

$$\tilde{J}^-(y_i) \;=\; \sum_{\nu=1}^{N} \sigma_{+\lambda}(y_i - z_\nu)\mu_\nu = 0 \quad , \quad \tilde{J}^0(y_i) \;=\; -\sum_{\nu=1}^{N} \xi(y_i, z_\nu)\mu_\nu \frac{\partial}{\partial \mu_\nu} = -\frac{\partial}{\partial y_i} + \frac{\partial}{\partial \lambda} \;\; .$$

When we plug these expressions into $S(y_i)$ we find

$$S(y_i) \;=\; \frac{\partial^2}{\partial y_i^2}\,, \tag{3.27}$$

just as on the sphere. In conclusion we have shown that the KZB equations lead to the following N differential equations for the WZNW N-point functions Ω^H

$$D_i^{\rm H} \Pi \Omega^H \;= 0\,,$$

$$\Omega^H(z_\nu, \mu_\nu, \lambda) \;=\; \left\langle \prod_{i=1}^{N} V_{j_i}(\mu_i|z_i) \right\rangle^{H}_{(\lambda,\tau)}\,, \tag{3.28}$$

$$D_i^{\rm H} \;= b^2 \frac{\partial^2}{\partial y_i^2} + \sum_{\nu=1}^{N} \left(\xi(y_i, z_\nu) \frac{\partial}{\partial z_\nu} - \Delta^H_{j_\nu} \partial_i \xi(y_i, z_\nu) \right) + 2\pi i \frac{\partial}{\partial \tau} \quad . \tag{3.29}$$

Let us recall that the derivatives $\partial/\partial z_\nu = \partial_\nu$ in $D_i^{\rm H}$ are still taken while keeping λ and μ_i fixed, in spite of the explicit appearance of derivatives with respect y_i.

In order to verify consistency of the two sets of equations with the proposed relation between Liouville and WZNW correlation functions, we rewrite the latter in the form

$$\Pi \Omega^H \;= |\eta(\tau)|^2 |\Theta_N'|^2 \Omega^L\,, \tag{3.30}$$

$$\Theta_N' \;= \theta'(0)^{\frac{N}{2b^2}} \prod_{\nu<\mu}^{N} \theta(z_\nu - z_\mu)^{\frac{1}{2b^2}} \prod_{i<j}^{N} \theta(y_i - y_j)^{\frac{1}{2b^2}} \prod_{\nu,i=1}^{N} \theta(z_\nu - y_i)^{-\frac{1}{2b^2}}\;. \tag{3.31}$$

In comparison to the earlier version, we have absorbed a factor into a re-definition of $|\Theta_N|^2$ and then expressed the new $|\Theta_N'|^2$ in terms of $|\theta|^2$ rather than F. Our result on the relation between correlation functions therefore implies

$$(\eta \Theta_N')^{-1} D_i^{\rm H} (\eta \Theta_N') \;=\; D_i^{\rm L} \quad . \tag{3.32}$$

This can be checked indeed by a lengthy but straightforward computation. In the process it is important to replace all the derivatives $\partial_\nu = \partial_\nu^H$ in the differential operators D_i^H through derivatives $\partial_\nu = \partial_\nu^L$ where the latter are taken while keeping y_i and u fixed. More precisely, we replace $\delta_i = \sum_\nu \xi(y_i, z_\nu)\partial_\nu^H$ in $D_i^{\rm H}$ by

$$\delta_i \;\equiv\; \sum_{\nu=1}^{N} \xi(y_i, z_\nu) \left(\partial_\nu^L + \partial_i \right) - \sum_{j \neq i} \xi(y_i, y_j) \left(\partial_i - \partial_j \right) \quad , \tag{3.33}$$

where ∂_i denote differentiation with respect to y_i, as before. It is easy to verify that δ_i satisfy $\delta_i \mu_\nu(y_i, z_\nu, u) = 0$ and $\delta_i \lambda(y_i, z_\nu, u) = 0$. Once this issue is taken care of, we can show that the differential operators indeed satisfy (3.32). The details of the computation are presented in appendix A.

4. Generalization to arbitrary genus

Equipped with the experience from genus $g = 0$ and $g = 1$, we now address the WZNW model on an arbitrary closed surface of genus g. Most of the analysis follows the ideas of previous sections, but the details require considerably more background concerning differentials on higher genus surfaces. We provide the most relevant details in appendix B. Since this section contains our main new result, we shall derive it first through our path integral arguments and then verify with the help of free field computations where both sides of the proposed relation possess the same residues. Comments on the relation between the KZB and BPZ differential equations are deferred to the concluding section.

4.1 Path integral derivation

Suppose we are given some compact Riemann surface Σ with moduli parametrized by the period matrix $\tau = (\tau_{ij})$. Since our fields are going to be multivalued as in the torus case, it is appropriate to pass to the universal cover $\tilde{\Sigma}$ of the surface right away. Using the famous Abel map, we embed $\tilde{\Sigma}$ into $\mathbb{C}^g$. From now on, we shall think of our fields as being defined on the image of the Abel map and hence consider them as functions of g complex coordinates $w_k, k = 1, \ldots, g$. In close analogy to the genus $g = 1$ case, we allow for nontrivial twists along the β-cycles, i.e.

$$\begin{aligned} \beta(w_k + \tau_{kl}n^l + m_k|\tau) &= e^{2\pi i n^l \lambda_l}\beta(w_k|\tau)\,, \\ \gamma(w_k + \tau_{kl}n^l + m_k|\tau) &= e^{-2\pi i n^l \lambda_l}\gamma(w_k|\tau)\,, \\ \phi(w_k + \tau_{kl}n^l + m_k|\tau) &= \phi(w_k|\tau) + \frac{2\pi n^l \mathrm{Im}\lambda_l}{b}\,. \end{aligned} \tag{4.1}$$

The complex parameter λ_k represents the twist along the β-cycle β_k. Thereby, we have introduced g complex parameters.

As in our discussion of the theory on the torus, spelling out the coupling of ϕ to the world-sheet curvature requires to split ϕ into a twisted zero mode ϕ_{sol} and a single valued fluctuation ϕ_q. The twisted zero mode ϕ_{sol} is now given by

$$\phi_{\mathrm{sol}} \;=\; \frac{2\pi}{b}\,\mathrm{Im}\lambda^k\;(\mathrm{Im}\tau)^{-1}_{kl}\;\mathrm{Im}w^l \;=\; \frac{2\pi}{b}\,\mathrm{Im}\lambda^k\;(\mathrm{Im}\tau)^{-1}_{kl}\;\mathrm{Im}\int_{w_0}^{w}\omega^l \tag{4.2}$$

where $\omega^l, l = 1.\ldots, g$, is a basis of holomorphic one-forms and indices l, k are raised and lowered with the trivial metric. The linear dilaton term couples the world-sheet curvature $\mathcal{R}$ to the doubly periodic fluctuation field $\phi_q = \phi - \phi_{\mathrm{sol}}$ in the same way as on the torus, see eq. (3.4).

So far, setting up the path integral for WZNW correlators on a surface of genus g was a straightforward extension of the torus case. But there is one important modification. As is well known, the λ-twisted differentials $\rho\beta$ and $\rho\bar{\beta}$ possess $g - 1$ zero modes. These give rise to $g - 1$ additional moduli if we decide to fix the value of the $\rho\beta$ and $\rho\bar{\beta}$ zero modes and to extend our path integral only over the remaining fluctuations. Our aim therefore is to compute the following N-point correlation functions

$$\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^{H}_{(\lambda,\varpi,\tau)} = \int \mathcal{D}^\lambda\phi\mathcal{D}^\lambda\gamma\tilde{\mathcal{D}}^\lambda\beta\ e^{-S[\phi,\gamma,\beta]}\prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu,|z_\nu)$$

over a Riemann surface with genus g. The symbol $\tilde{\mathcal{D}}^\lambda\beta$ reminds us not to integrate over β zero modes. We parametrize the latter by $g-1$ coordinates $\varpi = (\varpi_\sigma, \sigma = 1,\ldots,g-1)$ and place an explicit subscript ϖ on the correlator. The physical correlation functions may be recovered in principle through a finite dimensional integral over ϖ_σ.

Integration over γ leads to exactly the same expression (2.5) for the derivative of β as on the sphere and torus. But the corresponding β takes a different form. From the knowledge of its derivative and the boundary conditions (4.1) we may conclude that β must have the form

$$\beta(w) = \sum_{\nu=1}^{N}\mu_\nu\sigma_\lambda(w,z_\nu) + \sum_{\sigma=1}^{g-1}\varpi_\sigma\omega^\lambda_\sigma(w) \tag{4.3}$$

where ω^λ_σ denote a basis of λ twisted holomorphic differentials and the function $\sigma_\lambda(w,z)$ is the following differential,

$$\sigma_\lambda(w,z) = \frac{(h_\delta(w))^2}{\theta_\delta(\int_z^w\omega)}\frac{\theta_\delta(\lambda-\int_z^w\omega)}{\theta_\delta(\lambda)} \quad . \tag{4.4}$$

On the right hand side, we can use any odd spin structure δ. The θ function θ_δ and the 1/2-differential h_δ are defined in appendix B. Using properties of these objects it is possible to show that $\sigma_\lambda(z,w)$ has a simple pole at $z = w$ with residue $Res_{z=w}\sigma_\lambda(z,w) = 1$. Moreover the differential σ_λ satisfies the same periodic boundary condition as $\beta(w)$. Before we go on, let us point out that we had to fix the β zero mode in order to be able to reconstruct β from its derivative. Explicit formulas for the $g-1$ twisted holomorphic differentials can be found e.g. in [23].

The rest of our analysis proceeds essentially as before. A meromorphic one-differential with N poles is known to possess $N+2(g-1)$ zeroes. Hence, the analogue of the separation of variables formula (2.7) for a surface of genus $g \geq 1$ is given by

$$\rho(w)\beta(w) = u\,\frac{\prod_{i=1}^{N+2(g-1)}E(w,y_i)\sigma(w)^2}{\prod_{\nu=1}^{N}E(w,z_\nu)} =: u\,\rho(w)\,\mathcal{B}_g(y_i,z_\nu;w) \tag{4.5}$$

with y_i parametrizing the zeroes of $\rho\beta$. It seems that this formula has not appeared in the literature before. In order to reproduce the correct boundary conditions (4.1), y_i have to satisfy the condition

$$\lambda_l = \sum_{k=1}^{N+2(g-1)}\int_w^{y_k}\omega_l - \sum_{\nu=1}^{N}\int_w^{z_\nu}\omega_l - 2\int_{(g-1)w}^{\Delta}\omega_l\,, \tag{4.6}$$

which corresponds to the condition (3.9) for the genus one case. Notice that this condition does not depend on w. The function $\sigma(w)$ in eq. (4.5) is a $g/2$-differential without zeros nor poles. Its definition is reviewed is eq. (B.13) of appendix B along with the construction of the prime form E (see eq. (B.7)) and the Riemann class Δ. The function $\sigma(w)$ is needed in order for the right hand side of (4.5) to become a one-differential with the correct zeros and poles.

There is another point we would like to stress. Note that the equations (4.6) impose g constraints on the position of the zeroes y_i. Thereby, they define a $N+g-2$-dimensional hyper-surface in the configuration space of y_i. These hyper-surfaces sweep out the entire configuration space as we vary the g twist parameters. Changing the zero mode parameters ϖ_σ while keeping λ fixed also moves the position of zeroes y_i, but this motion takes place within the the hyper-surface defined by λ since eq. (4.6) is independent of ϖ. Hence, the infinitesimal changes of the twist parameters λ and the zero mode parameters ϖ span a $2g-1$ dimensional subspace of vectors tangent to the configuration space of y_i. Together with the shifts of the N light-cone momenta μ_ν, we thereby generate independent moves of all the zeroes y_i and of u. Let us also note that the light-cone momenta may be reconstructed from y_i according to

$$\rho(z_\nu)\mu_\nu \;=\; u\,\frac{\prod_{j=1}^{N+2(g-1)} E(z_\nu, y_j)\sigma(z_\nu)^2}{\prod_{\mu\neq\nu}^{N} E(z_\nu, z_\mu)}\;. \tag{4.7}$$

After these comments we can continue with our computation of WZNW correlators. Once the trivial integration over non-zero modes of β and $\bar\beta$ has been performed, we re-define the bosonic field,

$$\begin{aligned}&\varphi(w,\bar w) :=\\ &\qquad \phi(w,\bar w)+\frac{1}{2b}\left(\sum_{i=1}^{N+2(g-1)} \ln|E(w,y_i)|^2-\sum_{\nu=1}^{N}\ln|E(w,z_\nu)|^2+2\ln|\sigma(w)|^2-\ln|\rho(w)|^2\right).\end{aligned}$$

As on the torus, we may replace the multi-valued prime form E and σ by doubly periodic functions and thereby rewrite φ in terms of the single valued fluctuation field $\phi_q=\phi-\phi_{\rm sol}$,

$$\begin{aligned}&\varphi(w,\bar w) =\\ &\qquad \phi_q(w,\bar w)+\frac{1}{2b}\left(\sum_{i=1}^{N+2(g-1)} \ln F(w,y_i)-\sum_{\nu=1}^{N}\ln F(w,z_\nu)+2\ln H(w)-\ln|\rho(w)|^2+S_g\right).\end{aligned}$$

The functions $F(z), H(z)$ are defined in appendix B and the constant S_g is a shorthand for the following expression

$$\begin{aligned}S_g \;=& 2\pi\sum_i \mathrm{Im}\int_{w_0}^{y_i}\omega^l\;(\mathrm{Im}\tau)^{-1}_{lk}\;\mathrm{Im}\int_{w_0}^{y_i}\omega^k - 2\pi\sum_\nu \mathrm{Im}\int_{w_0}^{z_\nu}\omega^l\;(\mathrm{Im}\tau)^{-1}_{lk}\;\mathrm{Im}\int_{w_0}^{y_\nu}\omega^k-\\ &- 4\pi\mathrm{Im}\int_{(g-1)w_0}^{\Delta}\omega^l\;(\mathrm{Im}\tau)^{-1}_{lk}\;\mathrm{Im}\int_{(g-1)w_0}^{\Delta}\omega^k\,,\end{aligned}$$

which is independent of w_0. If we act with $\partial\bar\partial$ on φ we obtain the same expression (3.16) as on the torus. This uses that the contributions from the non-holomorphic part cancel each other. Furthermore, $\partial\bar\partial\ln\sigma(z)=0$ because σ has neither zeros nor poles.

After inserting the shift of variables from ϕ to φ, we can simplify the resulting expressions pretty much in the same way as for the torus case. Once more, the kinetic term for the field ϕ splits into the sum (3.14) of a constant term and a kinetic term for the fluctuation field ϕ_q. The former is given by

$$S_{\rm sol} = \frac{1}{2\pi}\int d^2w\, \bar\partial\phi_{\rm sol}\partial\phi_{\rm sol} = \frac{\pi}{b^2}{\rm Im}\lambda^l\,({\rm Im}\tau)^{-1}_{lk}\,{\rm Im}\lambda^k \quad . \tag{4.8}$$

A second auxiliary result concerns the contribution from the linear dilaton term. After the change of variables it provides us with a linear dilaton term for φ and the following additional terms,

$$\begin{aligned}\frac{Q_\phi}{8\pi}\int d^2w\frac{\sqrt{g}\mathcal{R}}{2b}&\left(\sum_{i=1}^{N+2(g-1)}\ln F(w,y_i)-\sum_{\nu=1}^{N}\ln F(w,z_\nu)+2\ln H(w)-\ln|\rho(w)|^2+S_g\right)\\ &=\sum_{i=1}^{N+2(g-1)}\ln(|\rho(y_i)|^{-1}H(y_i))-\sum_{\nu=1}^{N}\ln(|\rho(z_\nu)|^{-1}H(z_\nu))+(1-g)S_g+\frac{3}{2}U_g\,,\end{aligned}$$

where (see [24])

$$U_g \;=\; \frac{1}{192\pi^2}\int d^2wd^2y\sqrt{g(w)}\mathcal{R}(w)\sqrt{g(y)}\mathcal{R}(y)\ln F(w,y) \quad . \tag{4.9}$$

All these expressions can be verified using formulas from appendix B. Collecting all the above facts, our correlation function is given by

$$\left\langle\prod_{\nu=1}^{N}V_{j_\nu}(\mu_\nu|z_\nu)\right\rangle^{H}_{(\lambda,\varpi,\tau)} = \mathcal{C}\,|\Theta^g_N(u,y_i,z_\nu,\tau)|^2\left\langle\prod_{\nu=1}^{N}V_{\alpha_\nu}(z_\nu)\prod_{i=1}^{N+2(g-1)}V_{-1/2b}(y_i)\right\rangle^{L}_{\tau}$$

where the pre-factor Θ_N takes the form

$$\begin{aligned}|\Theta^g_N(u,y_i,z_\nu,\tau)|^2 &= (|u|^2e^{S_g})^{(1-g)}\prod_{i=1}^{N}H(z_i)^{-1-\frac{1}{b^2}}\prod_{k=1}^{N+2(g-1)}H(y_k)^{1+\frac{1}{b^2}}\times\\ &\times\prod_{r<s}^{N}F(z_r,z_s)^{\frac{1}{2b^2}}\prod_{k<l}^{N+2(g-1)}F(y_k,y_l)^{\frac{1}{2b^2}}\prod_{r=1}^{N}\prod_{k=1}^{N+2(g-1)}F(z_r,y_k)^{-\frac{1}{2b^2}},\end{aligned} \tag{4.10}$$

and we collected all the remaining terms in the quantity $\mathcal{C}$,

$$\mathcal{C} \;=\; e^{-S_{\rm sol}+(\frac{3}{2}+\frac{3}{4b^2})U_g}\,|\det\partial|^{-2}_\lambda \quad .$$

Before we conclude, let us observe that the constant $\mathcal{C}$ may be written as the ratio between partition function $Z^L_0 = Z^{\rm LD}_{Q_\varphi}$ of a linear dilation with background charge Q_φ and the product $Z^H_0 = |Z^{\beta\gamma}|^2 Z^{\rm LD}_{Q_\phi}$ where $Z^{\beta\gamma}$ is the partition function of a chiral $\beta\gamma$ system. In order to see this, we recall that the partition function in both H^+_3 model and Liouville field theory acquires it's leading (divergent) contribution from the asymptotic region where the interaction is negligible. Hence, we have

$$
\begin{aligned}
Z_0^H &= \int \mathcal{D}\phi_q \mathcal{D}^\lambda\gamma\mathcal{D}^\lambda\beta \; e^{-S_{\rm sol}-\frac{1}{2\pi}\int d^2w\left(\bar\partial\phi_q\partial\phi_q+\beta\bar\partial\gamma+\bar\beta\partial\bar\gamma+\frac{Q_\phi}{4}\sqrt{g}\mathcal{R}\,\phi_q\right)} = \\
&= |\det\partial|_\lambda^{-2} e^{-S_{\rm sol}} \int \mathcal{D}\phi_q e^{-\frac{1}{2\pi}\int d^2w\left(\bar\partial\phi_q\partial\phi_q+\frac{Q_\phi}{4}\sqrt{g}\mathcal{R}\,\phi_q\right)} \quad . \qquad (4.11)
\end{aligned}
$$

To further re-write the partition function Z_0^H, we must now change the background charge from Q_ϕ to Q_φ. We can achieve this using a result of [24] on correlation functions in a linear dilaton theory. When applied to the path integral in the formula for Z_0^H, it formally reads

$$
\begin{aligned}
\int \mathcal{D}\phi_q e^{-\frac{1}{2\pi}\int d^2w\left(\bar\partial_q\phi\partial\phi_q+\frac{Q_\phi}{4}\sqrt{g}\mathcal{R}\,\phi_q\right)} &= \frac{e^{-\frac{3}{4}Q^2U_g}}{|\det\partial|^2} = \\
= e^{-\frac{3}{4}(Q_\phi^2-Q_\varphi^2)U_g}\int\mathcal{D}\varphi e^{-\frac{1}{2\pi}\int d^2w\left(\bar\partial\varphi\partial\varphi+\frac{Q_\varphi}{4}\sqrt{g}\mathcal{R}\,\varphi\right)} &= e^{\left(\frac{3}{2}+\frac{3}{4b^2}\right)U_g} Z_0^L \qquad (4.12)
\end{aligned}
$$

where in the process of the calculation we inserted the explicit expression for the background charges Q_ϕ and Q_φ. Combining the previous two equations, we have shown,

$$
Z_0^H = |\det\partial|_\lambda^{-2} e^{-S_{\rm sol}+\left(\frac{3}{2}+\frac{3}{4b^2}\right)U_g}\; Z_0^L \quad . \qquad (4.13)
$$

In conclusion, our final result for the relation between normalized correlation functions in the H_3^+ model and in Liouville field theory reads

$$
\frac{1}{Z_0^H}\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu)\right\rangle^H_{(\lambda,\varpi,\tau)} = |\Theta_N^g(u,y_i,z_\nu,\tau)|^2\frac{1}{Z_0^L}\left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu)\prod_{i=1}^{N+2(g-1)} V_{-1/2b}(y_i)\right\rangle^L_\tau \qquad (4.14)
$$

where the function Θ_N^g is given by eq. (4.10). Let us also recall that the correlation functions of primaries (2.2) in the WZNW model at level $k=b^{-2}+2$ depend on the g twists λ_k and on $g-1$ zero modes ϖ_l in addition to the surface moduli. On the Liouville side, we compute the correlation functions of primaries (2.16) with a background charge $Q^L = Q_\varphi = b+1/b$ and with bulk cosmological constant $\mu_B = 4b^2$. The momenta α_ν are related to j_ν through eq. (2.14). The remaining momenta μ_ν in the WZNW model along with the $2g-1$ moduli λ_k and ϖ_l determine the insertion points y_i of $N+2(g-1)$ degenerate Liouville fields and a factor u that we absorbed in the definition of Θ_N^g. Finally, we stress that Z_0^H and Z_0^L are partition functions of free field theories. They agree with those of the H_3^+ and Liouville theory, respectively, if and only if we consider the theory on a surface of genus $g=1$. For higher genus, our relation (4.14) shows that the partition function Z^H of the H_3^+ model is related to a $2g-2$ point function in Liouville field theory.

4.2 Free field theory computations

In this subsection, we explain how to compute the residues of the first order poles in WZNW correlation function from free field theory (see also our discussion at the end of section 2).

We then determine the corresponding quantities for correlators in Liouville field theory and show that the results agree with our relation (4.14) for the full correlators.

As we have sketched in section 2.3 it is possible to compute the residues of poles in the H_3^+ correlation functions by inserting powers of the screening charges into correlators of a linear dilaton ϕ and a $\beta\gamma$ system. For the rest of this section we shall fix the world-sheet metric such that $\rho = 1$. The vertex operators and the usual screening charge take the same form as above

$$V_j(\mu|z) \equiv |\mu|^{2j+2} e^{\mu\gamma - \bar{\mu}\bar{\gamma}} e^{2b(j+1)\phi}, \quad S = \int d^2w S(w) = -\int d^2w \beta\bar{\beta} e^{2b\phi(w,\bar{w})} . \tag{4.15}$$

The N-point correlation function has a pole at $\sum_\nu (j_\nu + 1) = 1 - g - s$ ($s \in \mathbb{Z}_{\geq 0}$), whose residue is obtained by integrating the following correlators over the positions w_k

$$\begin{aligned}\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \prod_{k=1}^{s} S(w_k) \right\rangle &= \prod_{\nu=1}^{N} |\mu_\nu|^{2j_\nu+2} \left\langle \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi(w_k,\bar{w}_k)} \right\rangle \times \\ &\times \left\langle \prod_{\nu=1}^{N} e^{\mu_\nu\gamma(z_\nu)} \prod_{k=1}^{s} \beta(w_k) \right\rangle \left\langle \prod_{\nu=1}^{N} e^{-\bar{\mu}_\nu\bar{\gamma}(\bar{z}_\nu)} \prod_{k=1}^{s} [-\bar{\beta}(\bar{w}_k)] \right\rangle .\end{aligned} \tag{4.16}$$

Throughout this entire subsection, correlation functions are properly normalized such that the expectation value of the identity is trivial rather than the partition function. Since the free boson ϕ is subject to a background charge $Q = Q_\phi = b$, the contribution from the linear dilaton theory can be computed as

$$\left\langle \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi(w_k,\bar{w}_k)} \right\rangle = \Lambda_{\text{sol}} \left\langle \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi_q(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi_q(w_k,\bar{w}_k)} \right\rangle , \tag{4.17}$$

where the twisted zero mode of ϕ contributes the factor

$$\Lambda_{\text{sol}} = \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi_{\text{sol}}(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi_{\text{sol}}(w_k,\bar{w}_k)} . \tag{4.18}$$

The correlation function of the single valued fluctuation field $\phi_q = \phi - \phi_{\text{sol}}$ may be expressed through the functions F and H, see appendix B,

$$\begin{aligned}\left\langle \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi_q(z_\nu,\bar{z}_\nu)} \prod_{k=1}^{s} e^{2b\phi_q(w_k,\bar{w}_k)} \right\rangle &= \prod_{\nu=1}^{N} H(z_\nu)^{2b^2(j_\nu+1)} \prod_{k=1}^{s} H(w_k)^{2b^2} \times \\ \times \prod_{\nu<\mu}^{N} F(z_\nu.z_\mu)^{-2b^2(j_\nu+1)(j_\mu+1)} &\prod_{\nu=1}^{N}\prod_{k=1}^{s} F(z_\nu,w_k)^{-2b^2(j_\nu+1)} \prod_{k<l}^{s} F(w_k,w_l)^{-2b^2} .\end{aligned} \tag{4.19}$$

The factors in the second line of eq. (4.16) may be evaluated using the same formulas we employed in our path integral derivation. In particular, with the help of formula (4.5) we can conclude that

$$\left\langle \prod_{\nu=1}^{N} e^{\mu_\nu \gamma(z_\nu)} \prod_{k=1}^{s} \beta(w_k) \right\rangle = \prod_{k=1}^{s} \left[\sum_{\nu=1}^{N} \mu_\nu \sigma_\lambda(z_\nu, w_k) + \sum_{\sigma=1}^{g-1} \varpi_\sigma \omega_\sigma^\lambda \right]$$
$$= \prod_{k=1}^{s} \left[u \frac{\prod_{i=1}^{N+2(g-1)} E(y_i, w_k)\sigma(w_k)^2}{\prod_{\nu=1}^{N} E(z_\nu, w_k)} \right] .$$

Utilizing the relation between the prime form E and the special function F (see appendix B) we conclude

$$\left\langle \prod_{\nu=1}^{N} e^{\mu_\nu \gamma(z_\nu)} \prod_{k=1}^{s} \beta(w_k) \right\rangle \left\langle \prod_{\nu=1}^{N} e^{-\bar{\mu}_\nu \bar{\gamma}(\bar{z}_\nu)} \prod_{k=1}^{s} [-\bar{\beta}(\bar{w}_k)] \right\rangle$$
$$= |u|^{2s} \prod_{k=1}^{s} \left[\frac{\prod_{i=1}^{N+2(g-1)} |E(y_i, w_k)|^2 |\sigma(w_k)|^4}{\prod_{\nu=1}^{N} |E(z_\nu, w_k)|^2} \right] \tag{4.20}$$
$$= |u|^{2s} \prod_{k=1}^{s} e^{S_g - 2b\phi_{\mathrm{sol}}(w_k, \bar{w}_k)} \left[\frac{\prod_{i=1}^{N+2(g-1)} F(y_i, w_k) H(w_k)^2}{\prod_{\nu=1}^{N} F(z_\nu, w_k)} \right] .$$

Equations (4.18), (4.19) and (4.20) provide all the information we need in order to determine the residue (4.16) of the WZNW correlator at $\sum_\nu (j_\nu + 1) = 1 - g - s$.

We would like to rewrite the correlation functions in H_3^+ model in terms of Liouville theory. The background charge of Liouville theory is assumed to be $Q = b + 1/b$. We use the following vertex operators and Liouville screening charge,

$$V_\alpha(z, \bar{z}) = e^{2\alpha\varphi(z,\bar{z})} , \qquad S = \int d^2 w S(w) = \int d^2 w e^{2b\varphi(w,\bar{w})} \tag{4.21}$$

with $\alpha = b(j+1) + 1/(2b)$. According to the general result (4.14), it should be possible to reproduce the residues computed in the previous subsection from the expression

$$|\Theta_N^g|^2 \left\langle \prod_{\nu=1}^{N} e^{2\alpha_\nu \varphi(z_\nu, \bar{z}_\nu)} \prod_{i=1}^{N+2(g-1)} e^{-\frac{1}{b}\varphi(y_i, \bar{y}_i)} \prod_{k=1}^{s} e^{2b\varphi(w_k, \bar{w}_k)} \right\rangle \tag{4.22}$$

where Θ_N^g is given in eq. (4.10) and the correlation function is evaluated in a linear dilaton background with background charge $Q = Q_\varphi = b + 1/b$. Using the same formulas as in the previous subsection, we find

$$\left\langle \prod_{\nu=1}^{N} e^{2\alpha_\nu \varphi(z_\nu,\bar{z}_\nu)} \prod_{i=1}^{N+2(g-1)} e^{-\frac{1}{b}\varphi(y_i,\bar{y}_i)} \prod_{k=1}^{s} e^{2b\varphi(w_k,\bar{w}_k)} \right\rangle =$$

$$= \prod_{\nu<\mu}^{N} F(z_\nu, z_\mu)^{-2(b(j_\nu+1)+\frac{1}{2b})(b(j_\mu+1)+\frac{1}{2b})} \prod_{\nu=1}^{N} \prod_{i=1}^{N+2(g-1)} F(z_\nu, y_i)^{j_\nu+1+\frac{1}{2b^2}} \prod_{i<j}^{N+2(g-1)} F(y_i, y_j)^{-\frac{1}{2b^2}}$$

$$\times \prod_{\nu=1}^{N} \prod_{k=1}^{s} F(z_\nu, w_k)^{-2b^2(j_\nu+1)-1} \prod_{i=1}^{N+2(g-1)} \prod_{k=1}^{s} F(y_i, w_k) \prod_{k<l}^{s} F(w_k, w_l)^{-2b^2} \times$$

$$\times \prod_{\nu=1}^{N} H(z_\nu)^{2(b^2+1)(j_\nu+1)+1+\frac{1}{b^2}} \prod_{i=1}^{N+2(g-1)} H(y_i)^{-1-\frac{1}{b^2}} \prod_{k=1}^{s} H(w_k)^{2b^2+2} .$$

Our aim now is to replace the factors $F(z_\nu, y_i)^{j_\nu+1}$ which do neither appear in Θ_N^g nor in the residues of the WZNW model. Using the relation (4.7) with $\rho = 1$, we find

$$\prod_{\nu=1}^{N} \prod_{k=1}^{N+2(g-1)} E(z_\nu, y_k)^{j_\nu+1} \sigma(z_\nu)^{2j_\nu+2} \;=\; u^{s+g-1} \prod_{\nu=1}^{N} \mu^{j_\nu+1} \prod_{\mu<\rho}^{N} E(z_\mu, z_\rho)^{j_\mu+j_\rho+2} , \qquad (4.23)$$

or, equivalently,

$$\prod_{\nu=1}^{N} \prod_{k=1}^{N+2(g-1)} F(z_\nu, y_k)^{j_\nu+1} H(z_\nu)^{2j_\nu+2}$$
$$= (|u|^2 e^{S_g})^{s+g-1} \prod_{\nu=1}^{N} |\mu|^{2j_\nu+2} e^{2b(j_\nu+2)\phi_{\mathrm{sol}}(z_\nu,\bar{z}_\nu)} \prod_{\mu<\rho}^{N} F(z_\mu, z_\rho)^{j_\mu+j_\rho+2} . \qquad (4.24)$$

In deriving this equation we have made use of the equality $\sum_\nu (j_\nu + 1) = 1 - g - s$. Inserting (4.24) into the linear dilaton correlator we rewrite the residues of correlation functions in Liouville theory in an appropriate form. Once these are multiplied with our function Θ_N^g these reproduce the results of the previous subsection and thereby confirm nicely the outcome of our general path integral derivation.

5. Conclusions and outlook

In this work we proposed a new and elegant path integral derivation for the correspondence between local correlation functions in H_3^+ WZNW model and Liouville field theory. Our results reproduce the findings of [7] for correlators on the sphere and generalize them to surfaces of arbitrary genus g. Correlation functions of the H_3^+ WZNW model are determined from Liouville field theory through eq. (4.14). Physical correlators are obtained in the limit $\lambda_k \to 0$ of vanishing twist parameters and after integration over ϖ_σ. A correspondence between residues of correlation functions is encoded in eq. (4.14) and it was verified explicitly through free field calculations. For the torus, we have also explained how to map the BPZ equations for Liouville theory to the KZB equations in WZNW models. The extension of this analysis to higher genus was not addressed. We believe that an explicit comparison along the lines of section 3.2 is possible, though rather cumbersome.

It seems appropriate to add a few comments on our $2g-1$ moduli λ_k, ϖ_σ and to explain their relation with the twist parameters introduced by Bernard [23]. We recall that Bernard's constructions involve a $3g$-dimensional space of twists. It parametrizes a set of g group elements which determine boundary conditions of currents along the β-cycles. Our coordinates λ_k correspond to very special twists with group elements of the form $g_k \sim \exp(-2\pi i \lambda_k J^0)$ where J^0 is the Cartan generator. Consequently, insertions of the current $J^0(w)$ into correlators of the WZNW model may be converted into the action of some differential operator. The latter has exactly the same form as in Bernard's work (see eqs. (4.16) and (4.17) of [23]). For insertions of the component $J^-(w)$, the story is a bit different. In this case, we may use the relation $J^-(w) = \beta(w)$ along with our equation (4.3) to derive

$$\left\langle J^-(w) \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^H_{(\lambda,\varpi,\tau)} = \left(\sum_{\nu=1}^{N} \mu_\nu \sigma_\lambda(w, z_\nu) + \sum_{\sigma=1}^{g-1} \varpi_\sigma \omega_\sigma^\lambda(w) \right) \Omega^H_{(\lambda,\varpi,\tau)} \tag{5.1}$$

$$\text{where} \quad \Omega^H_{(\lambda,\varpi,\tau)} = \left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^H_{(\lambda,\varpi,\tau)} . \tag{5.2}$$

On the right hand side, the complex numbers ϖ_σ that multiply the holomorphic one-forms appear in place of differentiation with respect to twist parameters in Bernard's work. In this sense, our ϖ_σ are dual twist parameters. We are not convinced that insertions of the third current $J^+(w)$ can similarly be replaced by the action of some operator. Let us point out, however, that this is not crucial for a successful match between the BPZ and KZB-type differential equations. In fact, only certain combinations of the KZB equations appear in this context. The relevant ones emerge from inserting the Sugawara tensor at the points y_i at which J^- vanishes. Hence, the term J^+J^- drops out.

As explained e.g. in [25] (see also references therein), one of the ramifications of the geometric Langlands program involves conformal blocks of WZNW models at the so-called critical level and their relation with certain classical $\mathcal{W}$ algebras. In the case of the H_3^+ model, the critical level is $k = 2$. Hence, we reach this point in the limit $b \to \infty$ in which the associated Liouville theory becomes classical. For genus $g = 0$ correlation functions in the H_3^+-Liouville correspondence, the limit of infinite parameter b was analyzed in detail in [7]. It might be rewarding to carry out a similar investigation for surfaces of higher genus $g \neq 0$. In this case, the Gaudin Hamiltonians that emerge from the critical WZNW model on the sphere get replaced by Hamiltonians of Hitchin's integrable system.

The relation (4.14) between correlation functions in the H_3^+ model and Liouville field theory may be regarded as an 'off-critical' (and non-chiral) version of the geometric Langlands program. Let us recall that conformal blocks of the WZNW model diagonalize the action of the current algebra on the fusion product of its representation spaces in the same sense in which Clebsch-Gordan maps (block-) diagonalize the action of a Lie algebra on tensor products. When we are dealing with Lie algebras, the geometric Langlands program achieves more: it provides a distinguished basis in the tensor product consisting of eigenvectors of a classical $\mathcal{W}$-algebra. Once we go off-critical, the classical $\mathcal{W}$-algebra becomes quantum. In the case considered here, the $\mathcal{W}$-algebra is the Virasoro algebra. As usual, the action of the $\mathcal{W}$-algebra is block diagonalized by its conformal blocks. Putting all this together, an off-critical version of the geometric Langlands program should single out a distinguished basis for WZNW conformal blocks which may be expressed directly through conformal blocks of the $\mathcal{W}$-algebra. Our main result eq. (4.14) claims that for the H_3^+ model such a basis is given by the WZNW correlators on the left hand side. Let us stress that the proper basis is found for (twisted) correlators on any closed Riemann surface. It seems likely that a similar off-critical version of the geometric Langlands correspondence exists for other Lie algebras (see also comments below).

There are several extensions of our results that seem worthwhile being analyzed. To begin with, it would be interesting to study correlation functions on surfaces with boundaries. For the WZNW model, maximally symmetric boundary conditions were found in [26]. Using new boundary theories for $\beta\gamma$ systems (see [27]) along with ideas from a forthcoming paper on branes in the GL(1|1) model [28], a first order formulation for correlators with insertions of both bulk and boundary operators can be developed. An evaluation along the lines we presented above should then relate these to correlation functions in boundary Liouville theory [29 − 31]. For some disc amplitudes, such relations between correlators on a surface with boundary have been proposed in [32].

More importantly, it is very tempting to address generalizations to WZNW models of rank $r > 1$. First order formulations for models with higher rank are certainly known (see e.g. [33] and references therein) and it is likely that these may be employed to reduce correlators of WZNW primaries to correlation functions in conformal Toda theories. The evaluation of the corresponding WZNW path integral, however, requires significant new ideas, mainly because the nilpotent part of higher rank algebras is no longer abelian. This is directly linked to a non-linear dependence of the Kac-Wakimoto like action functionals on some of the fields γ. We plan to return to these issues in the near future.

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A. Differential equations for genus one case

In this appendix we would like to demonstrate that the differential operators (3.22) and (3.29) are related through equation (3.32). As we described in the main text, it is important to first replace the derivatives $\partial_\nu^H = \partial/\partial z_\nu$ which are evaluated for fixed μ_ν in terms of ∂_ν^L. This is achieved with the help of formula (3.33). In order to verify the latter, we must show that

$$\delta_i \mu_\nu = 0\,, \qquad \delta_i \lambda = 0\,. \tag{A.1}$$

The second equation can be shown trivially with eq. (3.9). The first property of δ_i may be established as follows. With eq. (3.9) we find

$$\begin{aligned}\mu_\nu^{-1}\delta_i\mu_\nu =& \xi(y_i,z_\nu)\Big(\sum_j \xi(z_\nu,y_j) - \sum_{\mu\neq\nu}\xi(z_\nu,z_\mu)\Big) + \sum_{\mu\neq\nu}\xi(y_i,z_\mu)\xi(z_\nu,z_\mu) - \\ &- \sum_\mu \xi(y_i,z_\mu)\xi(z_\nu,y_i) + \sum_{j\neq i}\xi(y_i,y_j)\left(\xi(z_\nu,y_i) - \xi(z_\nu,y_j)\right)\,.\end{aligned} \tag{A.2}$$

The right hand side is a double periodic function of y_i, which could become singular at $y_i \sim y_j, z_\nu$. We can analyze the singular behavior of $\mu_\nu^{-1}\delta_i\mu_\nu$ with the help of the following expansions,

$$\begin{aligned}\xi(z,z') =& \frac{1}{z-z'} - 2(z-z')\eta_1 + \mathcal{O}((z-z')^2)\,, \\ \partial_z\xi(z,z') =& -\frac{1}{(z-z')^2} - 2\eta_1 + \mathcal{O}(z-z')\,, \\ \xi(z,z')^2 =& \frac{1}{(z-z')^2} - 4\eta_1 + \mathcal{O}(z-z')\end{aligned} \tag{A.3}$$

at $z \sim z'$. In fact, using the above expansions for ξ one can show that $\mu_\nu^{-1}\delta_i\mu_\nu$ has no singularities when two of the insertion points y_i and z_ν approach each other. Since the whole expression is double periodic and free of singularities, the function should be constant, independent of y_i, z_ν. Therefore, it suffices to calculate it at one single point. Let us set $y_i = z_\nu$, then we find

$$\mu_\nu^{-1}\delta_i\mu_\nu = \sum_{\mu\neq\nu}\frac{\theta''(z_\nu - z_\mu)}{\theta(z_\nu - z_\mu)} - \sum_{j\neq i}\frac{\theta''(z_\nu - y_j)}{\theta(z_\nu - y_j)}\,. \tag{A.4}$$

Here, we have used the following expansion around $t \sim z_\nu$,

$$\frac{\theta'(t-z_\mu)}{\theta(t-z_\mu)} = \frac{\theta'(z_\nu-z_\mu)}{\theta(z_\nu-z_\mu)} + \left[\frac{\theta''(z_\nu-z_\mu)}{\theta(z_\nu-z_\mu)} - \left(\frac{\theta'(z_\nu-z_\mu)}{\theta(z_\nu-z_\mu)}\right)^2\right](t-z_\nu) + \mathcal{O}((t-z_\nu)^2)\,.$$

Furthermore, we can set each of y_j $(j \neq i)$ to be one of z_μ $(\mu \neq \nu)$ because the above quantity does not depend on y_j either. From the above equation we can see that the equation $\mu_\nu^{-1}\delta_i\mu_\nu = 0$ is indeed satisfied.

There appears another derivative in the differential operator D_i^{H} for torus correlation functions, namely the derivative ∂_τ with respect to the modulus. We would like to check that it acquires no corrections when we switch from the variables μ_ν to y_i, u, or, in other words,

$$\frac{\partial}{\partial\tau}\mu_\nu = 0\,. \tag{A.5}$$

In order to see this, we compute

$$4\pi i\mu_\nu^{-1}\frac{\partial}{\partial\tau}\mu_\nu \;=\; \sum_i \frac{\theta''(z_\nu-y_i)}{\theta(z_\nu-y_i)}+6\eta_1-\sum_{\mu\neq\nu}\frac{\theta''(z_\nu-z_\mu)}{\theta(z_\nu-z_\mu)}\;. \tag{A.6}$$

Here we should notice that

$$4\pi i\frac{\partial_\tau\theta(z)}{\theta(z)} = \frac{\theta''(z)}{\theta(z)} = \partial_z\xi(z,0)+\xi(z,0)^2\,, \qquad 4\pi i\frac{\partial_\tau\theta(z)}{\theta(z)} = -6\eta_1+\mathcal{O}(z)\;. \tag{A.7}$$

These formulas show that $\mu_\nu^{-1}\partial_\tau\mu_\nu$ has no singularities. As before we may evaluate the derivative at e.g. $y_i = z_i$ and then find indeed ${\mu_\nu}^{-1}\partial_\tau\mu_\nu = 0$.

In order to show (3.32), we have to understand the properties of the function Θ_N' that we defined in eq. (3.31),

$$\Theta_N' \;=\; \theta'(0)^{\frac{N}{2b^2}}\prod_{\nu<\mu}^{N}\theta(z_\nu-z_\mu)^{\frac{1}{2b^2}}\prod_{i<j}^{N}\theta(y_i-y_j)^{\frac{1}{2b^2}}\prod_{\nu,i=1}^{N}\theta(z_\nu-y_i)^{-\frac{1}{2b^2}}\;. \tag{A.8}$$

Conjugation of the various derivatives by the factor Θ_N' gives

$$\begin{aligned}
{\Theta_N'}^{-1}\frac{\partial}{\partial z_\nu}\Theta_N' &= \frac{\partial}{\partial z_\nu}+\frac{1}{2b^2}\sum_{\mu\neq\nu}\xi(z_\nu,z_\mu)-\frac{1}{2b^2}\sum_i\xi(z_\nu,y_i)\,,\\
{\Theta_N'}^{-1}\frac{\partial}{\partial y_i}\Theta_N' &= \frac{\partial}{\partial y_i}+\frac{1}{2b^2}\sum_{j\neq i}\xi(y_i,y_j)-\frac{1}{2b^2}\sum_\nu\xi(y_i,z_\nu)\,,\\
{\Theta_N'}^{-1}\frac{\partial}{\partial\tau}\Theta_N' &= \frac{\partial}{\partial\tau}+T\,,\\
T &= \frac{N}{2b^2}\frac{\partial_\tau\theta'(0)}{\theta'(0)}+\frac{1}{2b^2}\sum_{\nu<\mu}\frac{\partial_\tau\theta(z_\nu-z_\mu)}{\theta(z_\nu-z_\mu)}\\
&\quad+\frac{1}{2b^2}\sum_{i<j}\frac{\partial_\tau\theta(y_i-y_j)}{\theta(y_i-y_j)}-\frac{1}{2b^2}\sum_{\nu,i}\frac{\partial_\tau\theta(z_\nu-y_i)}{\theta(z_\nu-y_i)}\;.
\end{aligned}$$

Moreover, we note that

$$\begin{aligned}
{\Theta_N'}^{-1}b^2\frac{\partial^2}{\partial y_i^2}\Theta_N' &= b^2\frac{\partial^2}{\partial y_i^2}+\left(\sum_{j\neq i}\xi(y_i,y_j)-\sum_\nu\xi(y_i,z_\nu)\right)\frac{\partial}{\partial y_i}\\
&\quad+\frac{1}{2}\left(\sum_{j\neq i}\partial_i\xi(y_i,y_j)-\sum_\nu\partial_i\xi(y_i,z_\nu)\right)\\
&\quad+\frac{1}{4b^2}\left(\sum_{j\neq i}\xi(y_i,y_j)-\sum_\nu\xi(y_i,z_\nu)\right)^2\,,
\end{aligned} \tag{A.9}$$

and

$$\Theta_N'^{-1}\delta_i\Theta_N' = \sum_\nu \xi(y_i,z_\nu)\left(\frac{\partial}{\partial z_\nu}+\frac{1}{2b^2}\sum_{\mu\neq\nu}\xi(z_\nu,z_\mu)-\frac{1}{2b^2}\sum_j\xi(z_\nu,y_j)\right)+$$
$$+\sum_\nu \xi(y_i,z_\nu)\left(\frac{\partial}{\partial y_i}+\frac{1}{2b^2}\sum_{j\neq i}\xi(y_i,y_j)-\frac{1}{2b^2}\sum_\mu\xi(y_i,z_\mu)\right)-$$
$$-\sum_{j\neq i} \xi(y_i,y_j)\left(\frac{\partial}{\partial y_i}+\frac{1}{2b^2}\sum_{k\neq i}\xi(y_i,y_k)-\frac{1}{2b^2}\sum_\nu\xi(y_i,z_\nu)\right)+$$
$$+\sum_{j\neq i} \xi(y_i,y_j)\left(\frac{\partial}{\partial y_j}+\frac{1}{2b^2}\sum_{k\neq j}\xi(y_j,y_k)-\frac{1}{2b^2}\sum_\nu\xi(y_j,z_\nu)\right) .$$

This long list of equations puts us into the position to finally prove eq. (3.32),

$$\begin{aligned}(\Theta_N'\eta)^{-1}D_i^{\mathrm{H}}(\Theta_N'\eta)-D_i^{\mathrm{L}} =& -\frac{1}{4b^2}(\sum_{j\neq i}\xi(y_i,y_j)-\sum_\nu\xi(y_i,z_\nu))^2+\\ &+\frac{1}{2b^2}\sum_\nu\xi(y_i,z_\nu)(\sum_{\mu\neq\nu}\xi(z_\nu,z_\mu)-\sum_j\xi(z_\nu,y_j))+\\ &+\frac{1}{2b^2}\sum_{j\neq i}\xi(y_i,y_j)(\sum_{k\neq j}\xi(y_j,y_k)-\sum_\nu\xi(y_j,z_\nu))+\\ &+\frac{3}{2b^2}\eta_1-\frac{3}{4b^2}\sum_{j\neq i}\partial_i\xi(y_i,y_j)+\frac{1}{4b^2}\sum_\nu\partial_i\xi(y_i,z_\nu)+T .\end{aligned} \tag{A.10}$$

There could be double or single poles at $y_i = z_\nu, y_j$, but we can check that such terms are absent. Moreover, even if we regard the expression (A.10) as a function of y_j $(j\neq i)$, there are no singular terms. Therefore, the problem is whether the constant independent of y_i, y_j vanishes or not. We set $y_j = z_j$ for $j\neq i$ so that (A.10) becomes

$$(\Theta_N'\eta)^{-1}D_i^{\mathrm{H}}(\Theta_N'\eta)-D_i^{\mathrm{L}} = \frac{1}{4b^2}\left(\zeta(y_i,z_i)^2+\partial_i\zeta(y_i,z_i)\right)+\frac{3}{2b^2}\eta_1-\frac{1}{2b^2}\left(6\eta_1+\frac{\theta''(y_i-z_i)}{\theta(y_i-z_i)}\right), \tag{A.11}$$

where the last term of the right hand side comes from T and in the derivation we have also used

$$2\pi i\eta(\tau)^{-1}\frac{\partial}{\partial\tau}\eta(\tau) = -\eta_1 .$$

Taking the limit of $y_i\to z_i$, we indeed obtain the desired equation (3.32).

B. Theta functions on a general Riemann surface

In this appendix we summarize some basic results concerning a free boson on a Riemann surface of genus g. Basically, our exposition follows the discussions in [24]. See also [34–36].

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B.1 The prime form

We consider correlation functions on a compact Riemann surface Σ of genus g with a complex structure. Let us choose a canonical basis of homology cycles α_k, β_k $(k = 1, \cdots, g)$ satisfying

$$\oint_{\alpha_k} \omega_l = \delta_{kl} \, , \qquad \oint_{\beta_k} \omega_l = \tau_{kl} \, , \tag{B.1}$$

where $\omega_l (l = 1, \cdots, g)$ denote the holomorphic one-forms on Σ. The complex symmetric matrix τ_{kl} $(\mathrm{Im}\tau_{kl} > 0)$ is known as the period matrix. We now fix an arbitrary point p_0 in Σ and construct a map from the universal cover $\tilde{\Sigma}$ of the surface Σ to $\mathbb{C}^g$,

$$z_k(p) = \int_{p_0}^{p} \omega_k \tag{B.2}$$

with p, p_0 on $\tilde{\Sigma}$. This embedding of $\tilde{\Sigma}$ into $\mathbb{C}^g$ is known as the Abel map. Functions on the image of the Abel map descend to the surface Σ if they are periodic under all shifts of the form $z'_k = z_k + m_k + \tau_{kl} n^l$ with integer coefficients m_k, n^l. In order to construct a few basic objects on the surface Σ and its cover $\tilde{\Sigma}$, we recall the following definition of Riemann's theta function,

$$\theta_\delta(z|\tau) = \sum_{n \in \mathbb{Z}^g} \exp i\pi[(n+\delta_1)^k \tau_{kl} (n+\delta_1)^l + 2(n+\delta_1)^k (z+\delta_2)_k] \, , \tag{B.3}$$

where $\delta_k = (\delta_{1k}, \delta_{2k})$ with $\delta_{1k}, \delta_{2k} = 0, 1/2$ denotes the so-called *spin structure* along the α_k and β_k cycles. Under shifts along the $2g$ fundamental cycles, θ_δ behaves as

$$\theta_\delta(z + \tau n + m|\tau) = \exp[-i\pi(n^k \tau_{kl} n^l + 2n^k z_k)] \exp[2\pi i (\delta_1^k m_k - \delta_2^k n_k)] \theta_\delta(z|\tau) \; . \tag{B.4}$$

The Riemann vanishing theorem asserts that $\theta(z,\tau)$ vanishes in a point z on Σ if and only if there exists $g-1$ points p_i on Σ such that z can be written in the form

$$z = \Delta - \sum_{k=1}^{g-1} p_k \, , \tag{B.5}$$

where Δ is a fixed divisor on the surface Σ that is known as Riemann class. The right hand side of this equation could be considered as an element of $\mathbb{C}^g$ through application of the Abel map. Let us now introduce the following holomorphic 1/2-differential h_δ

$$(h_\delta(z))^2 = \sum_k \partial_k \theta_\delta(0|\tau) \omega_k(z) \; . \tag{B.6}$$

h_δ is the essential building block for the important prime form E

$$E(z,w) = \frac{\theta_\delta(\int_w^z \omega|\tau)}{h_\delta(z) h_\delta(w)} \tag{B.7}$$

which is defined for any odd spin structure δ. The prime form E has weight $(-1/2, 0) \times (-1/2, 0)$ and near its unique zero at $z = w$ one finds $E(z,w) \sim z - w$. Moreover, E is periodic under shifts z_l along the α_k-cycle as $z_l' = z_l + n_l$ with $n_l = \delta_{l,k}$. On the other hand, a non-trivial phase appears if we shift z_l along the β_k-cycle as $z_l' = z_l + \tau_{lk}$,

$$E(z + \tau_k, w) \;=\; -\exp\left(-i\pi\tau_{kk} - 2\pi \int_z^w \omega_k\right) E(z,w) \quad . \tag{B.8}$$

On the left hand side of this equation, the objects τ_k denotes the k^{th} column $\tau_{\cdot,k}$ of the period matrix τ_{lk}.

B.2 Free linear dilaton theory

Let us employ the prime form and some close relatives thereof to spell out the N-point functions of a free bosonic field with background charge Q. For the fluctuation around the zero mode, the correlation functions can be given as [24]

$$\left\langle \prod_{i=1}^N e^{2\alpha_i\varphi}(z_i) \right\rangle \;=\; \prod_{i<j} F(z_i,z_j)^{-2\alpha_i\alpha_j} \prod_i H(z_i)^{2Q\alpha_i} \quad , \tag{B.9}$$

where we assume that

$$\sum_i \alpha_i \;=\; Q(1-g) \quad . \tag{B.10}$$

We have defined F and H as

$$F(z,w) \;=\; \exp\left(-2\pi \mathrm{Im} \int_w^z \omega^k \; (\mathrm{Im}\tau)^{-1}_{kl} \; \mathrm{Im} \int_w^z \omega^l\right) |E(z,w)|^2 \quad , \tag{B.11}$$

and

$$H(z) \;=\; |\rho(z)| \exp\left(\frac{1}{16\pi}\int d^2w\sqrt{g}\mathcal{R}(w)\ln(F(z,w))\right) \quad . \tag{B.12}$$

Integration over the g holomorphic forms ω_k furnishes an element of $\mathbb{C}^g$ that can be multiplied with $(\mathrm{Im}\tau)^{-1}$. As in [24] we can rewrite the function H in a form

$$H(z) \;=\; \exp\left(\frac{2\pi}{g-1}\mathrm{Im}\int_{(g-1)z}^{\Delta} \omega^k \; (\mathrm{Im}\tau)^{-1}_{kl}\mathrm{Im} \int_{(g-1)z}^{\Delta} \omega^l\right) |\sigma(z)|^2 \tag{B.13}$$

which involves the Riemann class Δ that was introduced in the previous subsection. Let now p_k denote g arbitrary points on Σ. Then the function $\sigma(z)$ satisfies

$$\frac{\sigma(z)}{\sigma(w)} \;=\; \frac{\theta_0(z - \sum p_k + \Delta)}{\theta_0(w - \sum p_k + \Delta)} \prod_k \frac{E(w,p_k)}{E(z,p_k)} \quad . \tag{B.14}$$

It is important to mention that $\sigma(z)$ is a $g/2$-differential and that it has no zeros and poles. When translated with the k^{th} column vector of the period matrix, σ satisfies

$$\sigma(z+\tau_k, w) \;=\; \exp\left(\pi i(g-1)\tau_{kk} - 2\pi i \int_{(g-1)z}^{\Delta} \omega_k\right) \sigma(z,w) \quad . \tag{B.15}$$

As before, we combined the matrix elements τ_{lk} for $l = 1, \ldots, g$ into an element $\tau_k \in \mathbb{C}^g$.

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References

[1] J. Teschner, *On structure constants and fusion rules in the* $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ *WZNW model*, *Nucl. Phys.* **B 546** (1999) 390 [hep-th/9712256].

[2] J. Teschner, *The mini-superspace limit of the* $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ *WZNW model*, *Nucl. Phys.* **B 546** (1999) 369 [hep-th/9712258].

[3] J.M. Maldacena and H. Ooguri, *Strings in* AdS_3 *and the* $\mathrm{SL}(2,\mathbb{R})$ *WZW model. III: correlation functions*, *Phys. Rev.* **D 65** (2002) 106006 [hep-th/0111180].

[4] H. Dorn and H.J. Otto, *Two and three point functions in Liouville theory*, *Nucl. Phys.* **B 429** (1994) 375 [hep-th/9403141].

[5] A.B. Zamolodchikov and A.B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, *Nucl. Phys.* **B 477** (1996) 577 [hep-th/9506136].

[6] J. Teschner, *On the Liouville three point function*, *Phys. Lett.* **B 363** (1995) 65 [hep-th/9507109].

[7] S. Ribault and J. Teschner, H_3^+ *WZNW correlators from Liouville theory*, *JHEP* **06** (2005) 014 [hep-th/0502048].

[8] A.V. Stoyanovsky, *A relation between the Knizhnik-Zamolodchikov and Belavin-Polyakov-Zamolodchikov systems of partial differential equations*, math-ph/0012013.

[9] G. Giribet, *The string theory on* AdS_3 *as a marginal deformation of a linear dilaton background*, *Nucl. Phys.* **B 737** (2006) 209 [hep-th/0511252].

[10] G. Giribet and Y. Nakayama, *The Stoyanovsky-Ribault-Teschner map and string scattering amplitudes*, *Int. J. Mod. Phys.* **A 21** (2006) 4003 [hep-th/0505203].

[11] S. Nakamura and V. Niarchos, *Notes on the S-matrix of bosonic and topological non-critical strings*, *JHEP* **10** (2005) 025 [hep-th/0507252].

[12] T. Eguchi and H. Ooguri, *Conformal and current algebra on general Riemann surface*, *Nucl. Phys.* **B 282** (1987) 308.

[13] D. Bernard, *On the Wess-Zumino-Witten models on the torus*, *Nucl. Phys.* **B 303** (1988) 77.

[14] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on* AdS_3, *Adv. Theor. Math. Phys.* **2** (1998) 733 [hep-th/9806194].

[15] K. Hosomichi, K. Okuyama and Y. Satoh, *Free field approach to string theory on* AdS_3, *Nucl. Phys.* **B 598** (2001) 451 [hep-th/0009107].

[16] A.M. Polyakov, *Quantum geometry of bosonic strings*, *Phys. Lett.* **B 103** (1981) 207.

[17] E.K. Sklyanin, *Separation of variables in the Gaudin model*, *J. Sov. Math.* **47** (1989) 2473 [*Zap. Nauchn. Semin.* **164** (1987) 151].

[18] V. Schomerus, *Non-compact string backgrounds and non-rational CFT*, *Phys. Rept.* **431** (2006) 39, [hep-th/0509155].

[19] G. Giribet and C. Nuñez, *Correlators in* AdS_3 *string theory*, *JHEP* **06** (2001) 010, [hep-th/0105200].

[20] B. Enriquez, B. Feigin and V. Rubtsov, *Separation of variables for Gaudin-Calogero systems*, q-alg/9605030.

[21] G. Felder and A. Schorr, *Separation of variables for quantum integrable systems on elliptic curves*, math.QA/9905072.

[22] K. Gawedzki, *Noncompact WZW conformal field theories*, hep-th/9110076.

[23] D. Bernard, *On the Wess-Zumino-Witten models on Riemann surfaces*, *Nucl. Phys.* **B 309** (1988) 145.

[24] E.P. Verlinde and H.L. Verlinde, *Chiral bosonization, determinants and the string partition function*, *Nucl. Phys.* **B 288** (1987) 357.

[25] E. Frenkel, *Ramifications of the geometric Langlands program*, math.QA/0611294.

[26] B. Ponsot, V. Schomerus and J. Teschner, *Branes in the euclidean AdS_3*, *JHEP* **02** (2002) 016 [hep-th/0112198].

[27] T. Creutzig, T. Quella and V. Schomerus, *New boundary conditions for the $c = -2$ ghost system*, hep-th/0612040.

[28] T. Creutzig, T. Quella and V. Schomerus, *Branes in the* GL(1|1) *WZNW-Model*, arXiv:0708.0583.

[29] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory. I: boundary state and boundary two-point function*, hep-th/0001012.

[30] J. Teschner, *Remarks on Liouville theory with boundary*, hep-th/0009138.

[31] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152.

[32] K. Hosomichi and S. Ribault, *Solution of the H_3^+ model on a disc*, *JHEP* **01** (2007) 057 [hep-th/0610117].

[33] J.L. Petersen, J. Rasmussen and M. Yu, *Free field realizations of 2D current algebras, screening currents and primary fields*, *Nucl. Phys.* **B 502** (1997) 649 [hep-th/9704052].

[34] L. Alvarez-Gaume, G. W. Moore and C. Vafa, *Theta functions, modular invariance, and strings*, *Commun. Math. Phys.* **106** (1986) 1.

[35] J. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics **352**, Springer-Verlag (1973).

[36] D. Mumford, *Tata lectures on Theta, Vols. I, II*, Progress in Mathematics **43**, Birkhäuser (1984).

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